Real Time Modeling of Cardiac Tissue

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- We need to model 2D and 3D hearts

- Reproduce recordings used by machines (Electrode and ECGs)
What is the ECG

Atria

Ventricles

ECG

Using optical mapping.
Rabbit heart
ECG leads.
Things to Note:

Direction of activation gives QRS (given by Purkinje activations)

T-wave is given by the irregular wave back
Multi dimensional system (from 0 to 3D)

Open heart and Purkinje network
Multi dimensional system (from 0 to 3D)

Open heart and Purkinje network
T-wave is given by the irregular wave back

How do we implement ECG?
where \( u_i \) and \( u_e \) are the intra- and extracellular myocardial potentials at point \( r \) and time \( t \), respectively. Let the current densities be of the form \( \mathbf{j}_i = -g_i \nabla u_i \) for the intra-, extracellular, and extracardiac regions, respectively. There are no current sources or sinks within the body, so the continuity equation requires

\[
\begin{align*}
0 &= \nabla \cdot (\mathbf{j}_i + \mathbf{j}_e) \Big|_{r \in \Omega_H} \\
0 &= \nabla \cdot \mathbf{j}_0 \Big|_{r \in \Omega_0},
\end{align*}
\]

And the flux continuity across the boundary between the heart and extracardiac medium requires

\[
\begin{align*}
u_e &= u_i \\
\mathbf{j}_0 \cdot \mathbf{n} &= (\mathbf{j}_i + \mathbf{j}_e) \cdot \mathbf{n}
\end{align*}
\]

along the boundary \( \partial \Omega_H \). Within the heart, transmembrane potential differences \( V_m (r, t) \) provide an equivalent cardiac source when related as

\[
\mathbf{j} (r) = -g_i \nabla V_m,
\]

where \( g_i \) is the intracellular membrane conductance. We may then express the total current density as a sum including both the transmembrane potential \( V_m \) and the total electrostatic potential \( \varphi (r, t) \)

\[
\mathbf{j} = -\sigma_0 \nabla \varphi - g_i \nabla V_m.
\]

Since the divergence of the total current density is zero according to equation 4,

\[
0 = -\nabla \cdot (\sigma_0 \nabla \varphi) - \nabla (g_i \nabla V_m),
\]

It is possible to write a Poisson equation for the electrostatic potential in terms of the transmembrane potential

\[
\nabla^2 \varphi (r) = -\frac{g_i}{\sigma_0} \nabla^2 V_m.
\]

Since the ECG probe is located external to the heart, \( r \notin \Omega_H \). By the Neumann boundary conditions imposed upon the Green’s function, the surface term is zero if we take the approximation that the conducting medium has equal anisotropy ratios, or \( g_e \propto g_i \). For an infinite and homogeneous extracardiac medium \( \Omega_0 \), we have the Green’s function

\[
G (r; r') = \frac{1}{4\pi |r - r'|}.
\]

Substitution of this Green’s function, assuming equal anisotropy ratios, and utilizing the divergence theorem, we arrive at the integral formulation for the electrostatic potential at point \( r \) in terms of the transmembrane potential

\[
\varphi (r) = \frac{g_i}{4\pi \sigma_0} \int_{\Omega_H} d^3 r' \frac{\nabla^2 V_m (r')}{|r - r'|}.
\]

Since the dimensional scales associated with the transmembrane potential difference and ECG amplitude are known, the important characteristic involved in OM-ECG calculation is the relationship between its amplitude and time. Assuming the transmembrane potential has the form \( V_m (r) = V_m (x, y) \), with constant \( V_m \) along the \( z \)-direction, we have the proportionality

\[
\varphi (r) \propto \int d^2 r' \frac{\nabla^2 V_m (x, y)}{|r - r'|}.
\]
Reconstructed ECG (from experiment, or numerical)

\[ \varphi (\mathbf{r}) = \frac{g_i}{4\pi\sigma_0} \int_{\Omega_H} d^3r' \frac{\nabla'^2 V_m (\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \]
- Reconstructed ECG (from experiment, or numerical)

\[
\varphi (r) = \frac{g_i}{4\pi \sigma_0} \int_{\Omega_H} d^3 r' \frac{\nabla'^2 V_m (r')}{|r - r'|}.
\]

Advantage!
Real time Simulations of this:

- Ventricular Tachycardia
- Ventricular Fibrillation
- Multiple spiral waves
- Single spiral waves
Simulations

2D simulations

- Multiple spiral waves
- Single spiral waves

Mathematically, each current can be modeled by an ODE:

\[ \frac{dV}{dt} = \sum I_i \]

where

\[ I_i = g_i(V - E_i) \]

and

\[ g_i = f(V, t) \]

The more complex the model the more equations to solve.
Simulations

Stiff ODEs

Euler Method

Time for the upstroke $\sim 5$ milliseconds!!

i.e. 1 second requires 10,000 iterations!
Simulations

1 second requires 10,000 iterations!

Each cell (4 to 24) ODEs

Number of cells in tissue? Millions!

have to Solve: $10 \times 10^{11}$ ODEs per second