Lagrangian Reachabililty

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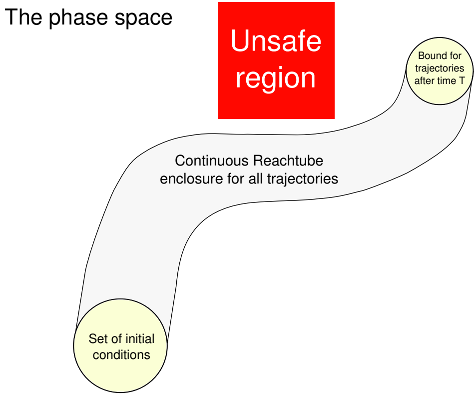
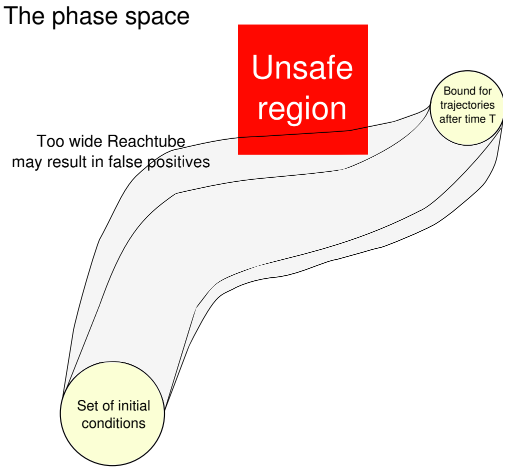
Problem statement

Given a continuous dynamical system (set of ODEs)
Compute continuous Reachtube overestimate bounding the ODE trajectories



Problem statement 2

Too wide Reachtube may result in false positives.



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C. Fan, J. Kapinski, X. Jin, and S. Mitra. *Locally optimal reach set overapproximation for nonlinear systems.* In Proceedings of the 13th International Conference on Embedded Software, EMSOFT '16, pages 6:1–6:10, New York, NY, USA, 2016. ACM.

Our approach

Our approach:

Explore possibilities of computation of the Reachtube overestimate using the Lagrangian coordinates method and identify advantages of doing so.

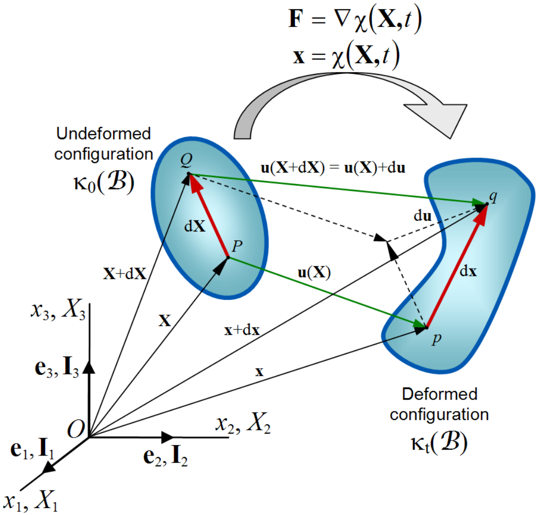

Figure: general idea of Lagrangian (deformed) coordinates source: wikipedia.org

We work with time-variant ordinary differential equations (ODEs):

$$x'(t) = F(t, x(t)), \tag{1a}$$

$$x(t_0) = x_0, \tag{1b}$$

where $x\colon \mathbb{R} \to \mathbb{R}^n$. We assume that F is a smooth function, which guarantees short-time existence of solutions.

For t_0, set of states $\mathcal{X} \subset \mathbb{R}^n$, and $T > t_0$, our goal is to compute a sequence of time-stamped sets of states $(R_1, t_1), \ldots, (R_k, t_k)$ satisfying (tight):

$$\mathrm{Reach}\left(\left(t_0, \mathcal{X}\right), \left[t_{i-1}, t_i\right]\right) \subset R_i \text{ for } i = 1, \ldots, k,$$

where $\mathrm{Reach}\left(\left(t_0, \mathcal{X}\right), \left[t_{i-1}, t_i\right]\right)$ denotes the set of reachable states of (1) in the interval $[t_{i-1}, t_i]$.

How our algorithm works?

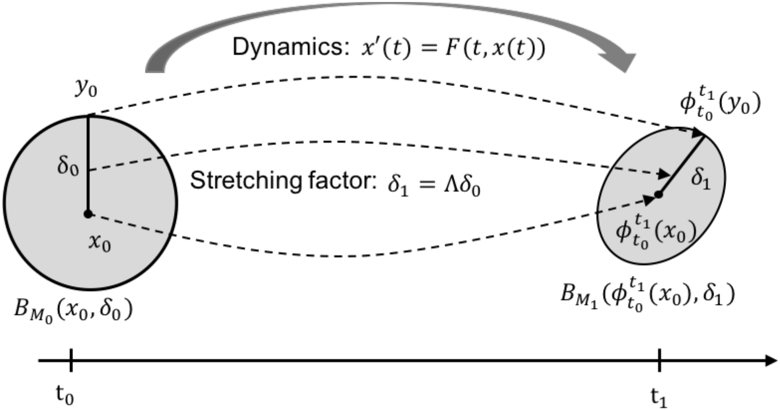

Figure: Illustration of our algorithm

General idea of the algorithm

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1. Compute over-approximation of the *gradient of the solution-flows*, and the *Cauchy-Green deformation tensor* (currently done by CAPD library),
2. Optimize for positive-definite symmetric matrix M_1, defining the weighted norm minimizing the Streching Factor for a choice of gradient,
3. Compute an upper bound for the Stretching Factor Λ, then the ball-overestimate is

$$B_{M_1}(\phi_{t_0}^{t_1}(x_0), \Lambda \cdot \delta_0).$$

Details 1

Cauchy-Green deformation tensor:

$$CGDT = (\nabla_X x)^T \cdot \nabla_X x. \tag{2}$$

Stretching factor

$$\sqrt{\lambda_{max}(CGDT)} > 0 \tag{3}$$

Definition

Given positive-definite symmetric matrix $M \in \mathbb{R}^{n\times n}$ we define the *M-norm* of $\mathbb{R}^n$ vectors by

$$\|y\|_M = \sqrt{y^T M y}. \tag{4}$$

Given the decomposition

$$M = C^T C,$$

the matrix norm inducted by (4) is

$$\|A\|_M = \sqrt{\lambda_{max}\left((C^T)^{-1} \cdot A^T \cdot M \cdot A \cdot C^{-1}\right)}, \tag{5}$$

Notation

Let $\phi_{t_0}^{t_1}(x)$ be the solution of (1) with the initial condition (t_0, x) at time t_1,
let $D_x\phi_{t_0}^{t_1}$ be the gradient of the flow.
Let $M_0, M_1 \in \mathbb{R}^{n\times n}$ be positive-definite symmetric matrices, and $C_0^T C_0 = M_0$, $C_1^T C_1 = M_1$ be their decompositions respectively.

Theory 1

Theorem

Let $\mathcal{X} = B_{M_0}(x_0, \delta_0) \subset \mathbb{R}^n$ *be a set of initial states Assume that there exists a compact, conservative enclosure* $\mathcal{D} \subset \mathbb{R}^{n \times n}$ *for the gradients, such that:*

$$D_x \phi_{t_0}^{t_1}(x) \in \mathcal{D} \text{ for all } x \in \mathcal{X}. \tag{6}$$

Suppose $\Lambda > 0$ *is such that:*

$$\Lambda \geq \sqrt{\lambda_{max}\left((C_0^T)^{-1} D^T M_1 D C_0^{-1}\right)}, \text{ for all } D \in \mathcal{D}.$$

Then it holds that:

$$\phi_{t_0}^{t_1}(x) \in B_{M_1}(\phi_{t_0}^{t_1}(x_0), \Lambda \cdot \delta_0).$$

Theory 2

Theorem (LRT-Conservativity)

Assume that the rigorous tool used in the Lagrangian Reachtube Algorithm (LRT) produces conservative gradient enclosures for system (1)*, and it guarantees existence of the solutions within time intervals. Assume also that the LRT terminates on the provided inputs.*

Then, the output of the LRT is a conservative reachtube over-approximation of (1) at times $\{t_j\}_{j=0}^{k}$, that is:

$$((t_0, \mathcal{X}), t_j) \subset \underline{B_{M_j}([x_j], \delta_j)}, \text{ for } j = 1, \ldots, k,$$

bounded solutions exists for all intermediate times $t \in (t_j, t_{j+1})$.

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1. To compute an over-approximation of the gradient we use *CAPD* library (C^1 Lohner algorithm),
2. We implemented an interface to communicate

$$\text{MATLAB} \leftrightarrow \text{C++},$$

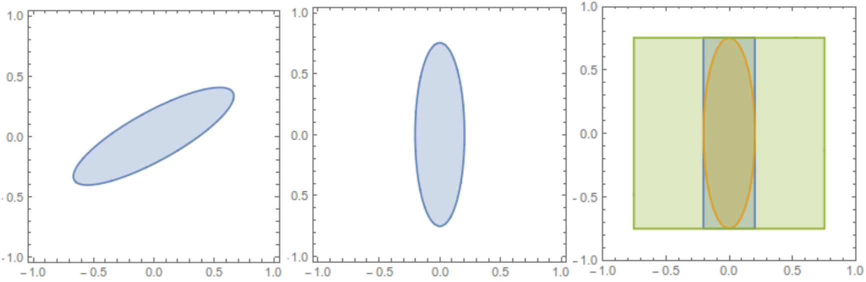
   we optimize for positive definite matrix M_1 using a Matlab package, and compute bound for the eigenvalues of the deformation tensor matrices using Intlab,

Algorithm instability

BM	dt	T	ID	LRT		Flow*		(direct) CAPD		nLRT	
				(F/I)V	(A/I)V	(F/I)V	(A/I)V	(F/I)V	(A/I)V	(F/I)V	(A/I)V
B(2)	0.01	20	0.02	**7.7e-5**	**0.09**	7.9e-5	0.15	Fail	Fail	Fail	Fail
I(2)	0.01	20	0.02	**4.3e-9**	**0.09**	5e-9	0.12	7.4e-9	0.10	1.45	1.31
L(3)	0.001	2	1.4e-6	1.6e5	9.0e3	2.1e13	7.1e12	**4.5e4**	**1.7e3**	1.2e22	4.9e19
F(2)	0.01	40	2e-3	**1.2e-42**	1.01	NA	NA	6.86e-4	**0.18**	5.64e7	3.21e5
M(2)	1e-3	4	2e-3	**0.006**	**0.22**	0.29	0.54	0.29	0.52	4.01	2.24
R(4)	0.01	20	1e-2	**2.2e-19**	**0.07**	2.4e-15	0.31	2.9e-11	0.23	Fail	Fail
O(7)	0.01	4	2e-4	**71.08**	**34.35**	272	5.1e4	4.3e3	620	Fail	Fail
P(12))	5e-4	0.1	0.01	**5.25**	**4.76**	290	64.6	280	62.4	19.4	6.2

Figure: Set of benchmarks from [1] + Lorenz equations L(3), Mitchell Schaeffer cardiac cell model M(2)

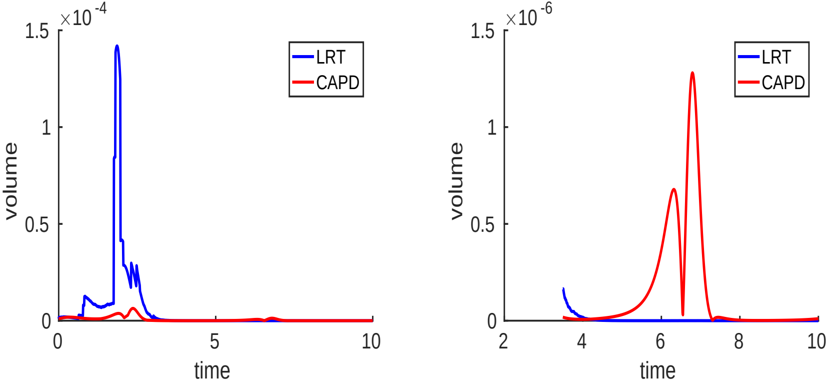
Wrapping effect



((a)) ((b)) ((c))

Figure: (a) A ball in the weighted norm given by M of radius 1 (the ellipsoidal set). (b) The ellipsoidal set in its eigen-coordinates (unrotated). (c) Volume computation by wrapping the ellipsoidal set in a box: blue rectangle in eigen-coordinates and green square in canonical-coordinates.

Furture work



(a) time horizon $= [0, 10]$ (b) time horizon $= [3.5, 10]$

Figure: Volume comparison for forced Van der Pol Oscillator

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- Apparently, for ODEs exhibiting asymptotically stable behaviour, LRT results in tighter Reachtube over-approximation,
- Some improvements of the algorithm are still possible,
- Future goal – Reachtube enclosures for PDEs.