Temporal Logic as Filtering

(Published in HSCC 2016)

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CyberCardia Project Meeting
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Motivation

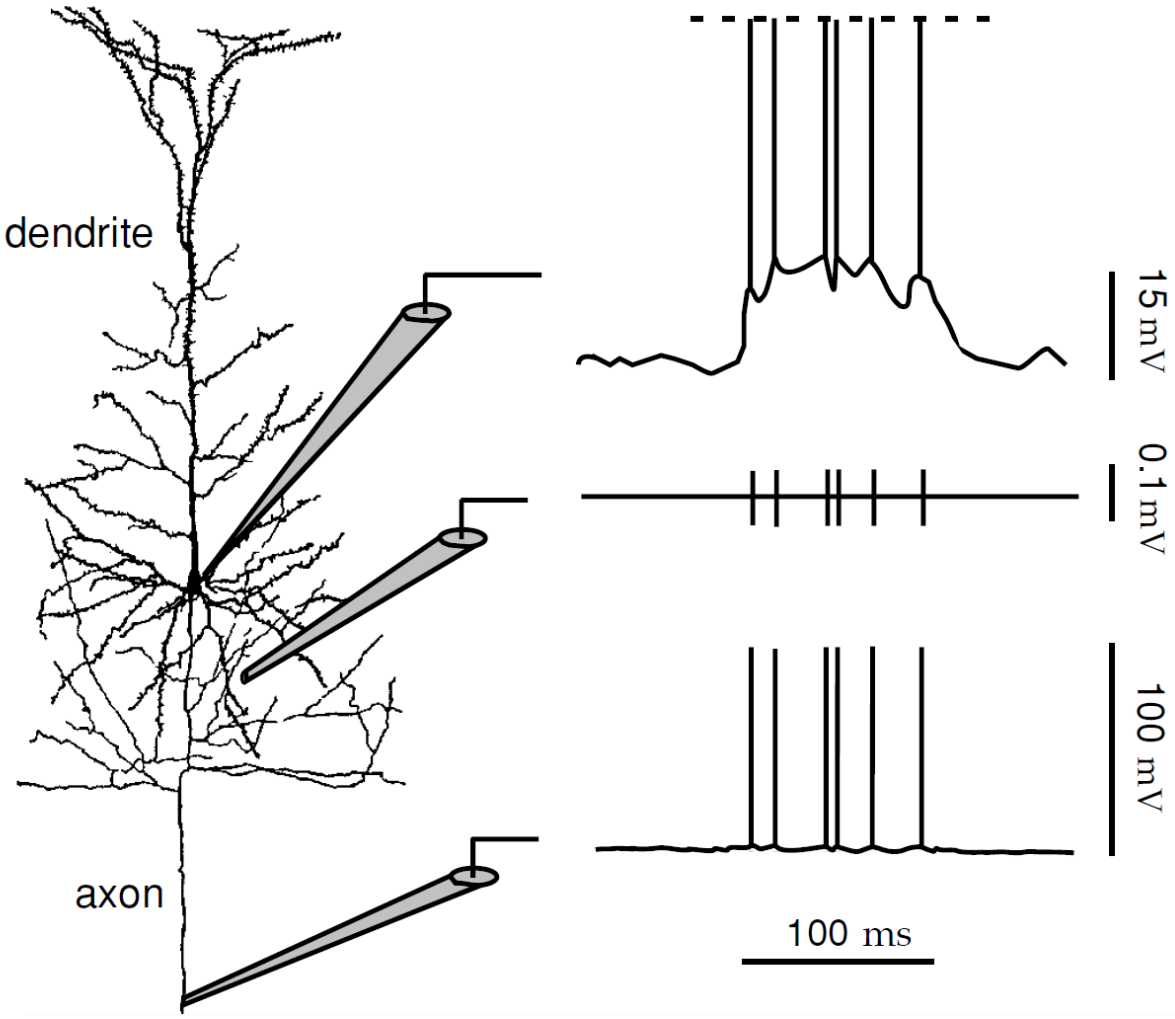
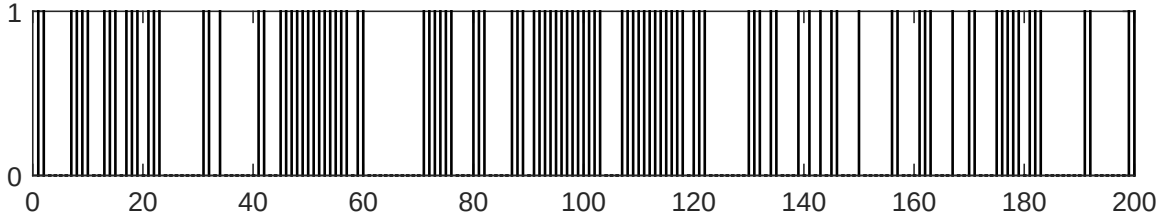


Figure: Neuron recordings

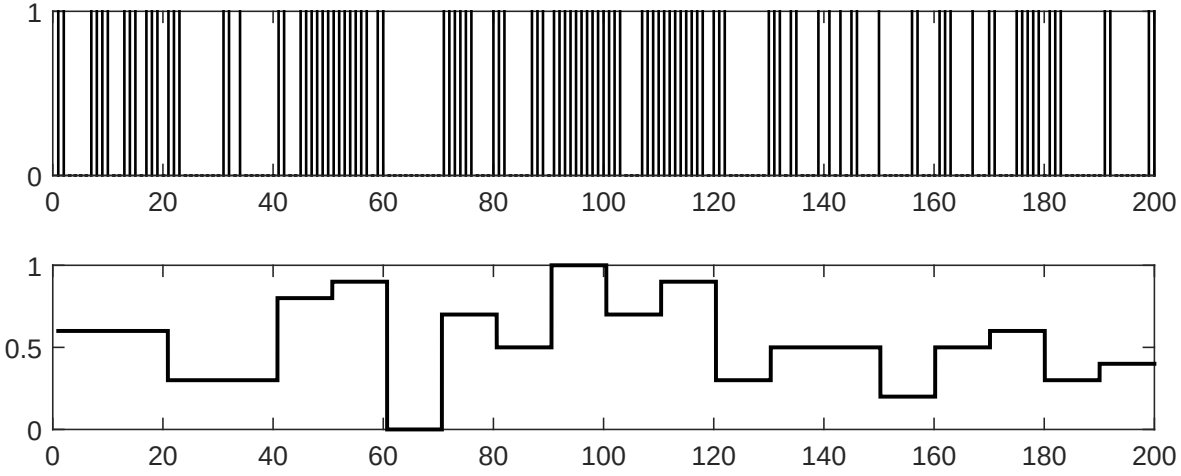
[1] *P. Dayan, L.F. Abbott. Theoretical Neuroscience, The MIT Press, 2001*

Motivation



Spike sequence

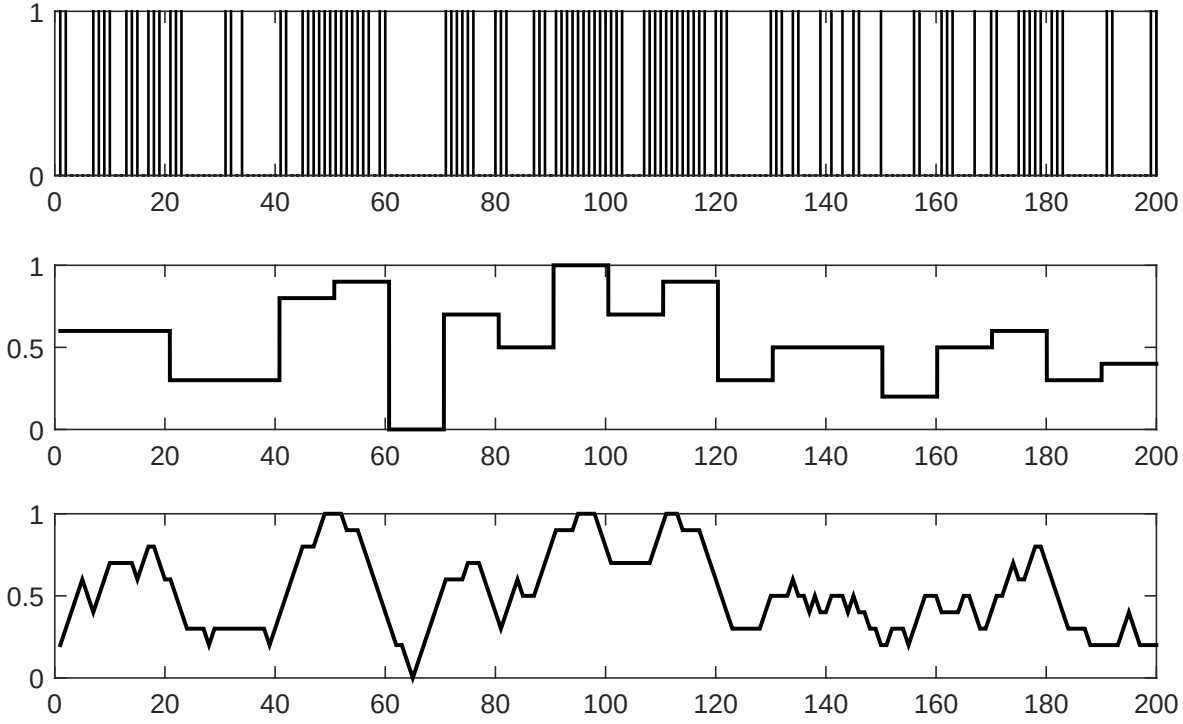
Motivation



Spike sequence

Spike rate
(Binning)

Motivation



Spike sequence

Spike rate (Binning)

Spike rate (Window sliding)

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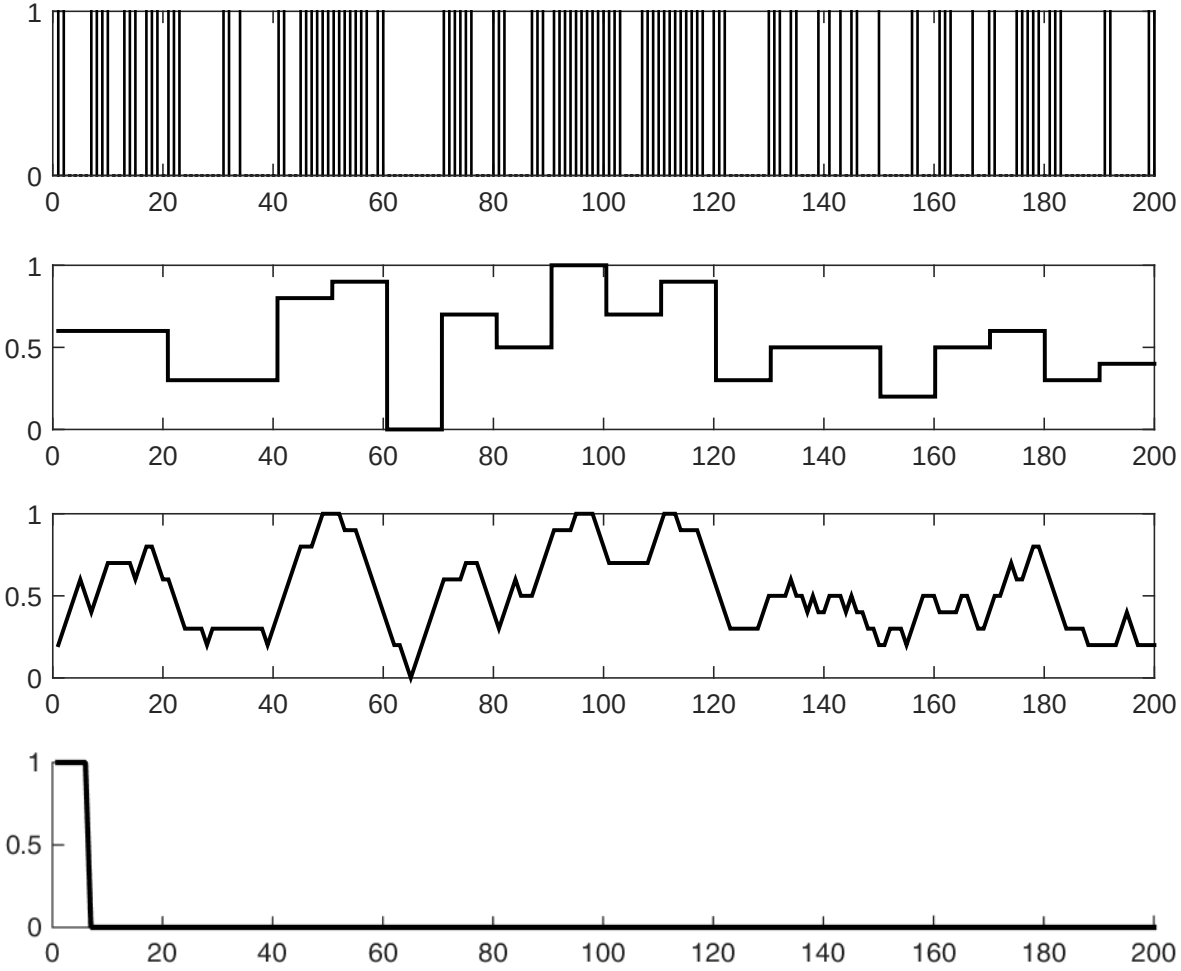


Spike sequence

Spike rate (Binning)

Spike rate (Window sliding)

Filtering: Convolution

$$conv(t) = (\rho * w)(t) = \sum_{i=0}^{T} \rho(i) w(t-i),$$

$$w(t) = \begin{cases} \frac{1}{\Delta t}, & \text{if } t \in [-\frac{\Delta t}{2}; \frac{\Delta t}{2}] \\ 0, & \text{otherwise} \end{cases}$$

Figure: Convolution with a rectangular window

MTL: Metric Temporal Logic

➤ *MTL Syntax*

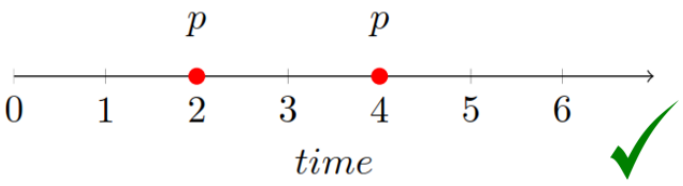
$$\Phi = \{\, p \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi\, \mathcal{U}_I\, \psi \mid \varphi\, S_I\, \psi \}$$

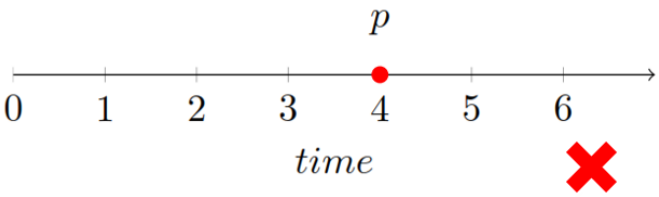
- $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$
- Finally operator: $\Diamond_I\, \varphi = \top\, \mathcal{U}_I \varphi$
- Globally operator: $\Box_I\, \varphi = \neg \Diamond_I \neg\varphi$
- Analogously for past operators Once $\diamondminus_I$ and Historically $\boxminus_I$

[4] R. Koymans, Specifying Real-Time Properties with Metric Temporal Logic, 1990

MTL: Metric Temporal Logic

Example. $(x, 0) \models \Diamond_{[0,2]} p$





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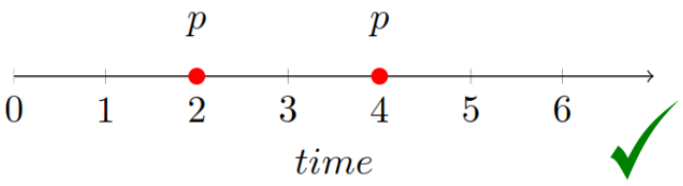
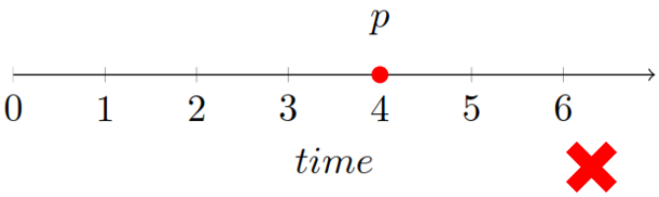
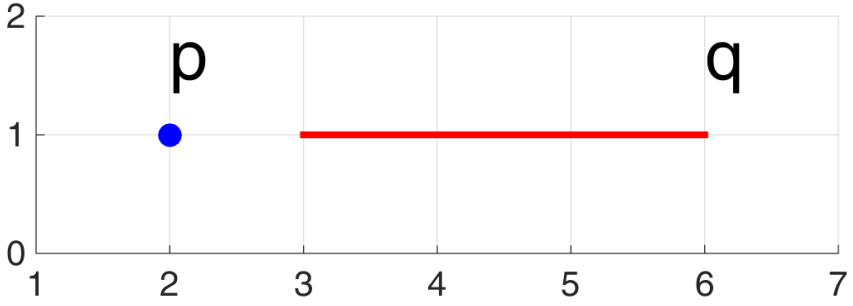
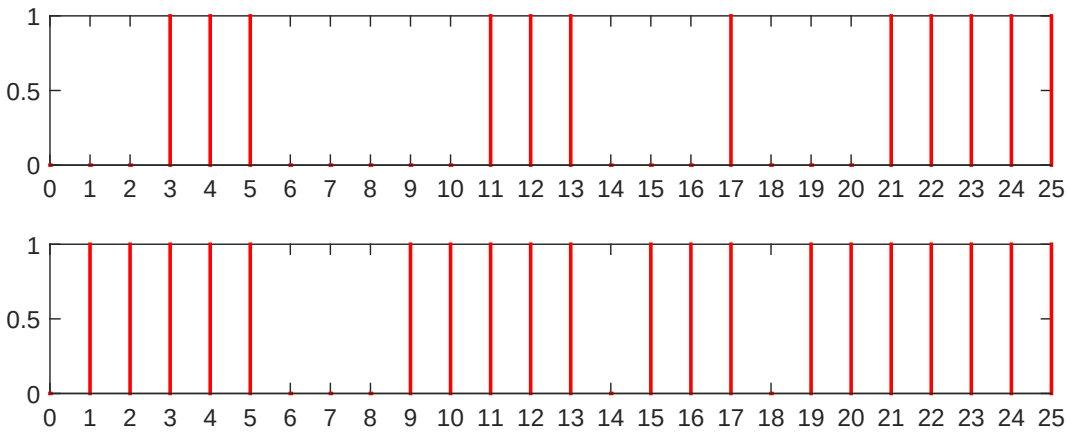




Example. $(x, 0) \models \Diamond_{[2,4]} p \wedge \Box_{[3,6]} q$



MTL: Standard Semantics



Discrete signal:
$(x, t) \models p$

Finally:
$(x, t) \models \Diamond_{[0,2]} p$

$$(x, t) \models \Diamond_I \varphi \iff \exists j \in (t + I) \cap \mathbb{T} : (x, j) \models \varphi$$

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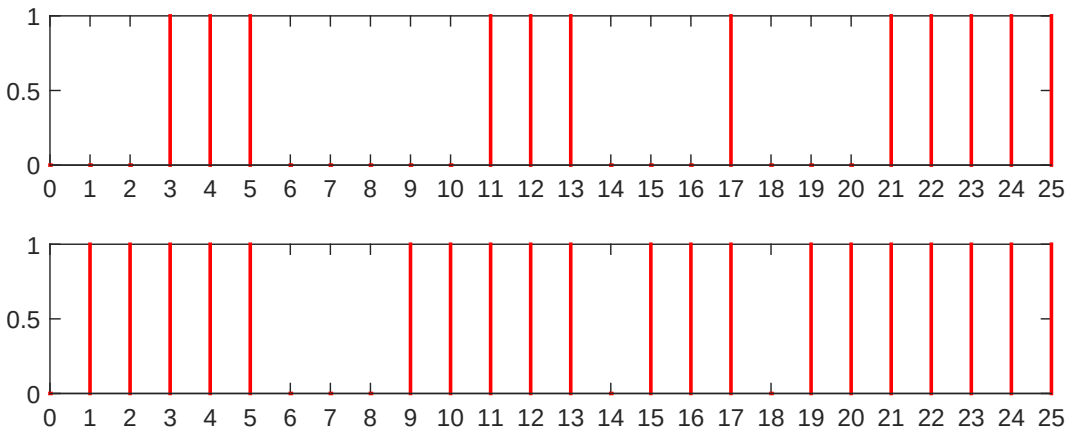

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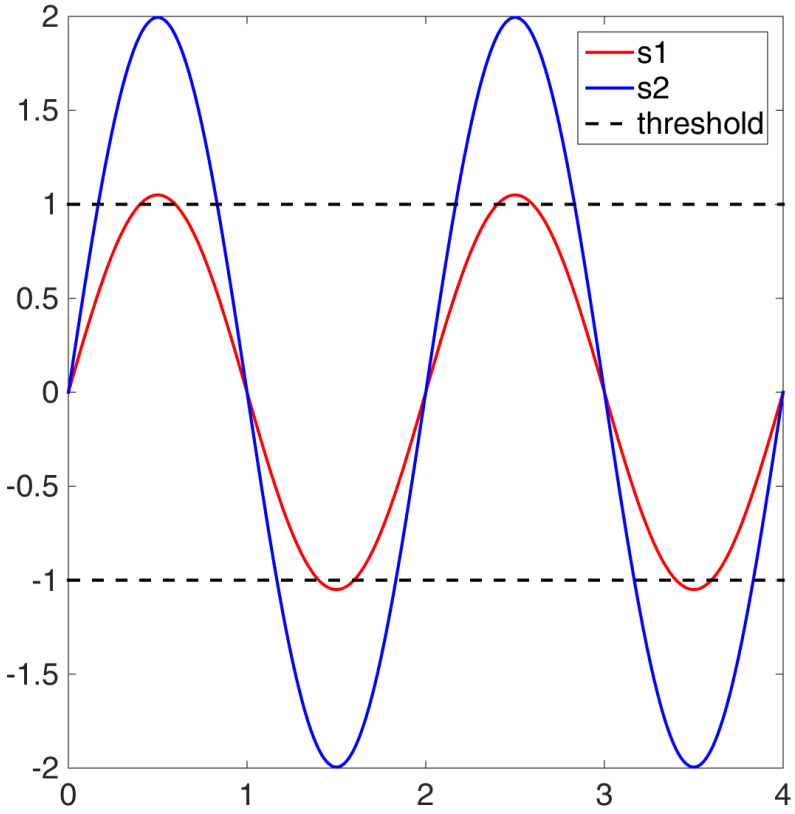
- ☞ Standard Boolean MTL semantics is qualitative
- ☞ It does not provide any quantitative measure

MTL: Robust Semantics

$\varphi =$ *"If the value of the signal drops below -1, then it should also raise above +1 within two time units."*



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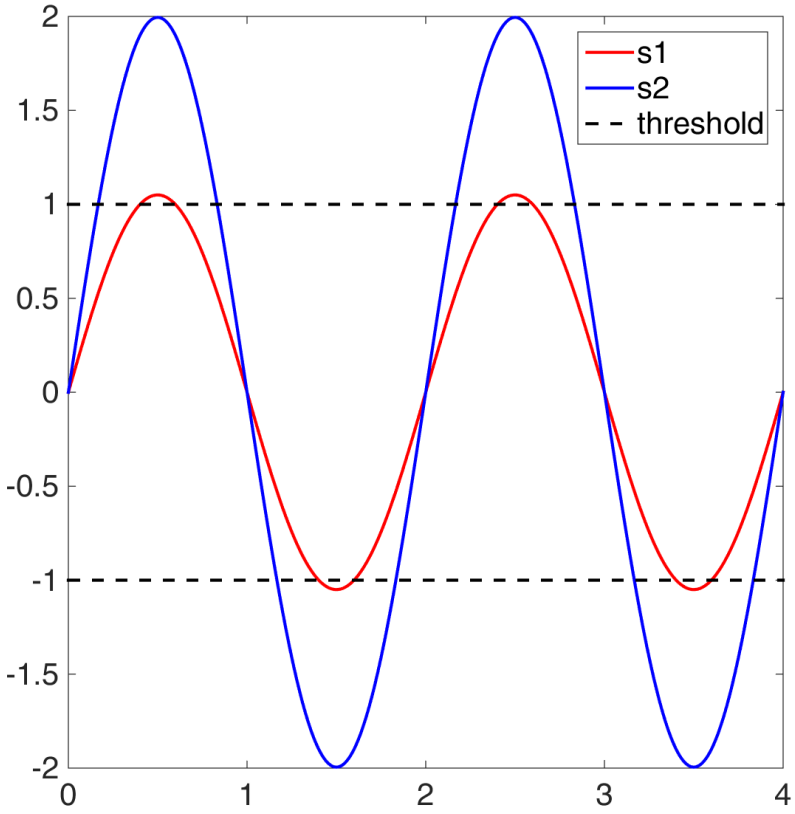

$$\Box \left(p_1 \longrightarrow \Diamond_{\leq 2} p_2\right)$$

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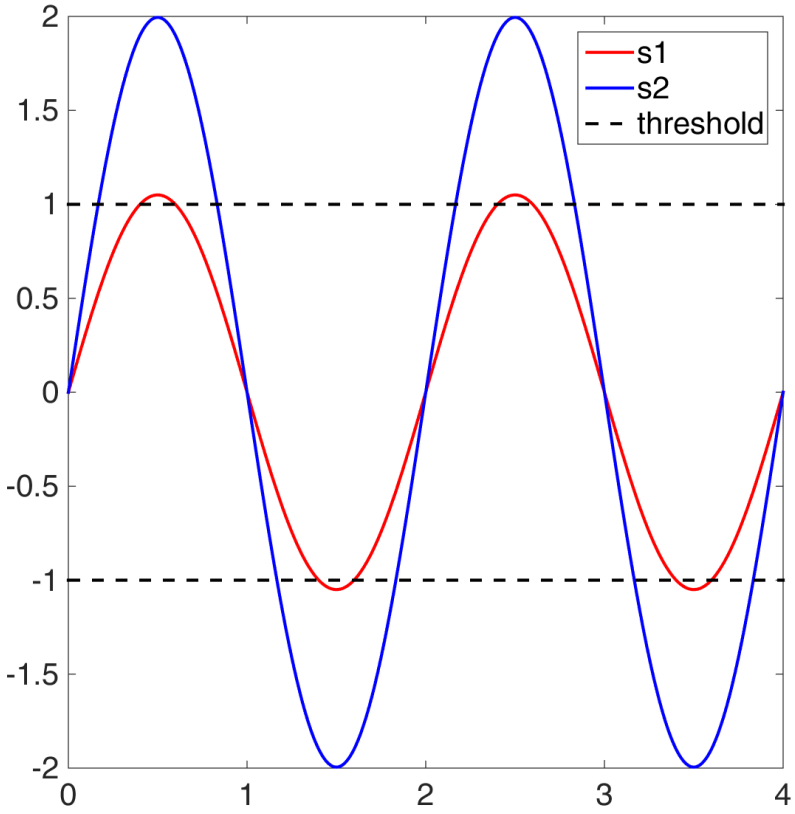

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☞ Robustness degree is the bound on the perturbation that the signal can tolerate without changing the truth value of a specification

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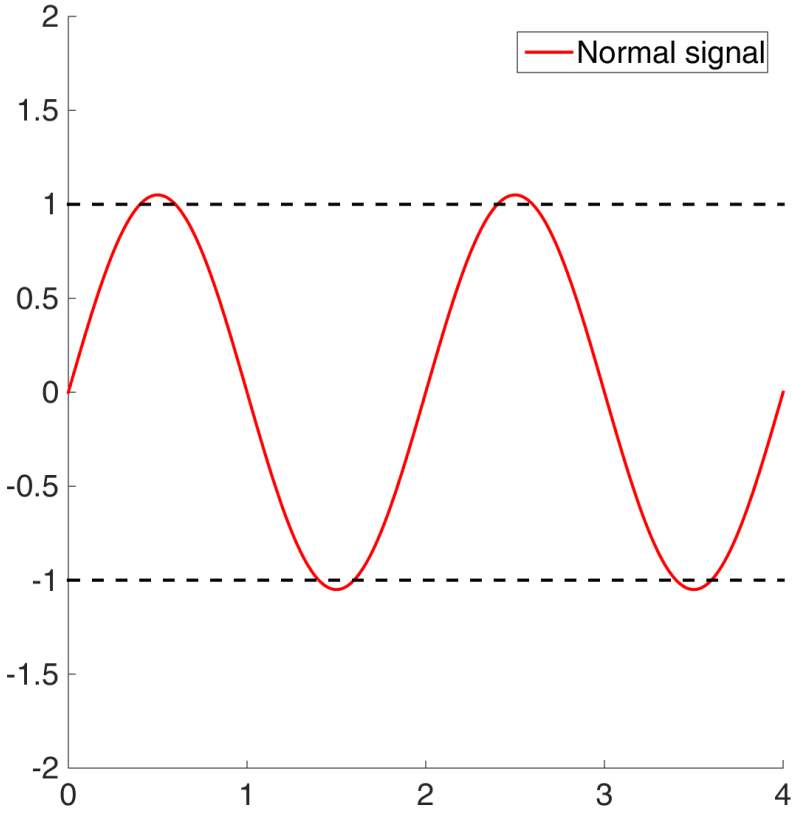

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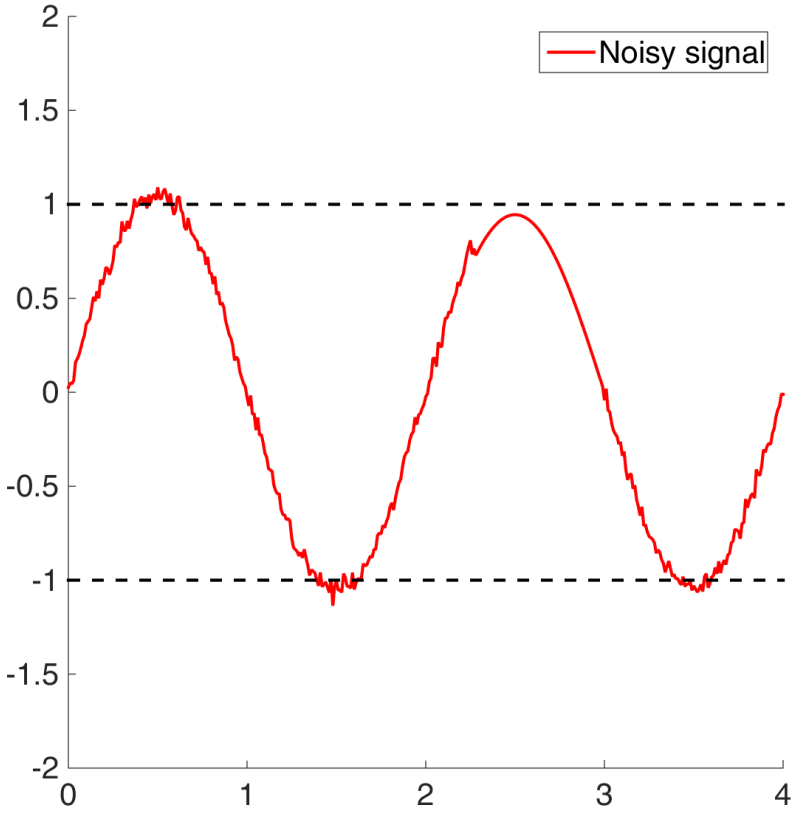

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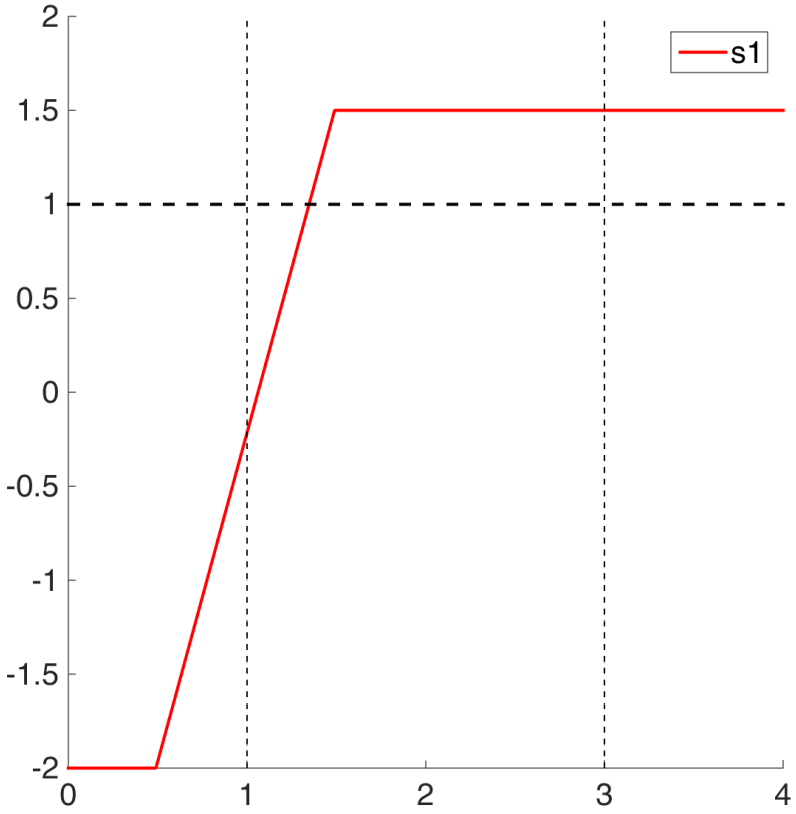
MTL: Time Robustness



$$\varphi = \Diamond_{[1,3]}(s \geq 1)$$

- Time robustness indicates the effect on satisfaction of shifting events in time

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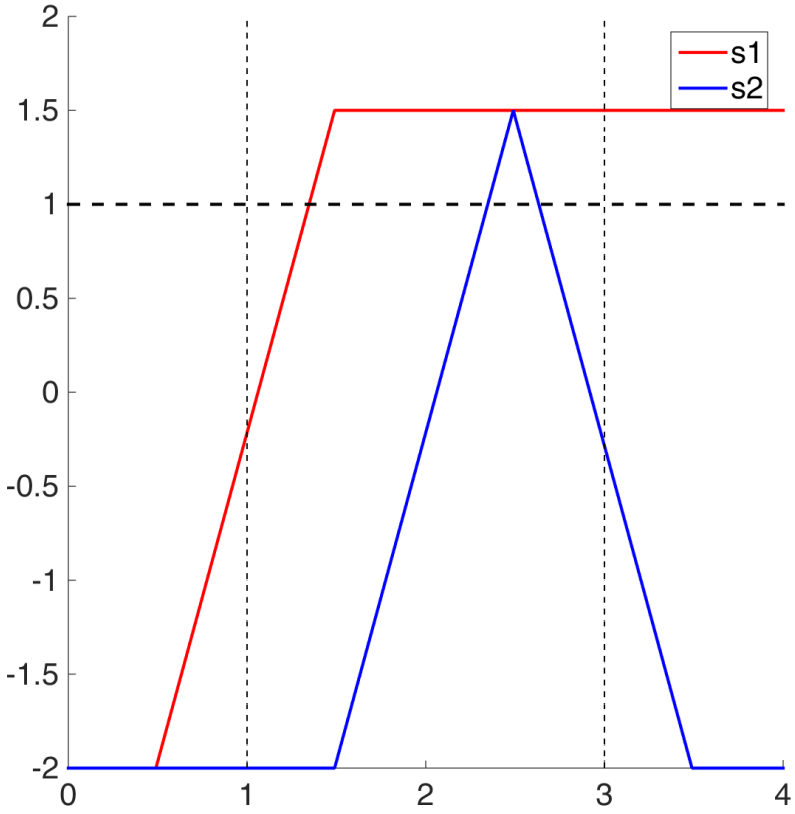

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MTL: Quantitative Semantics

Example. "In a next 10 years You will finally get a Turing award..."

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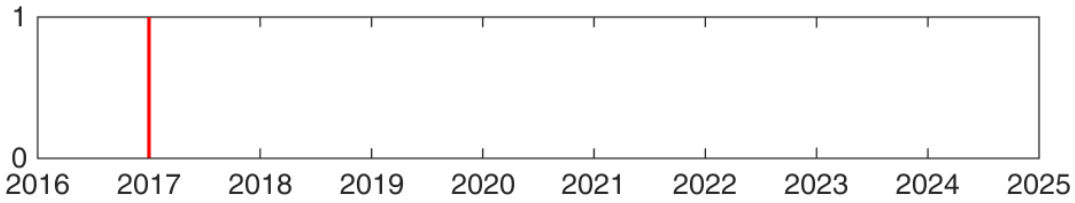

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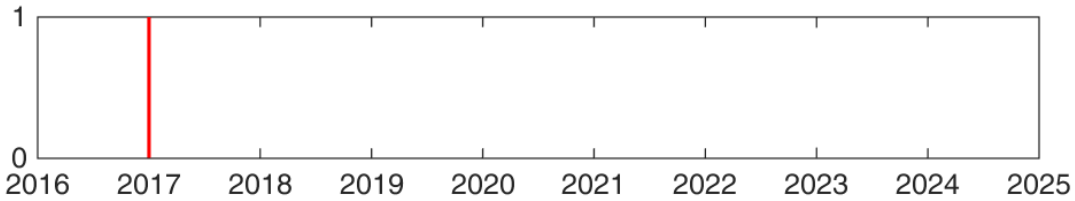

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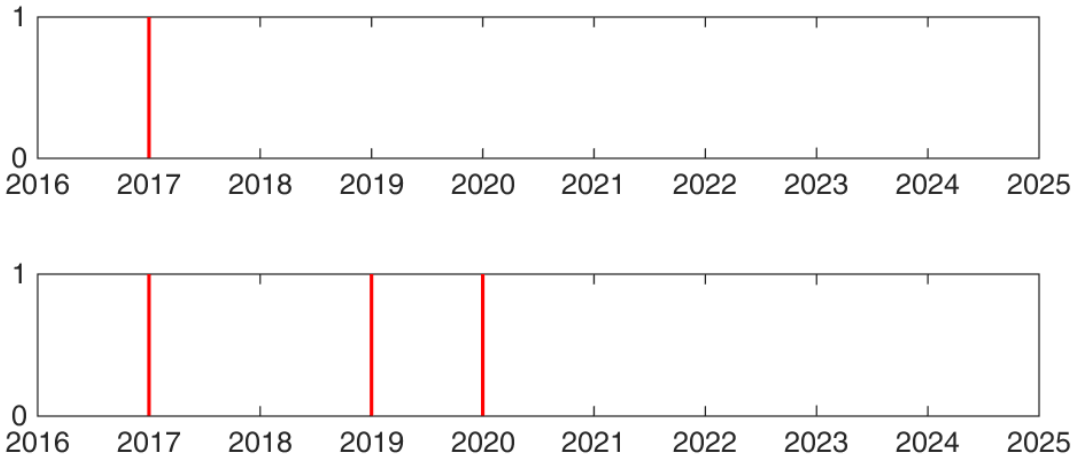

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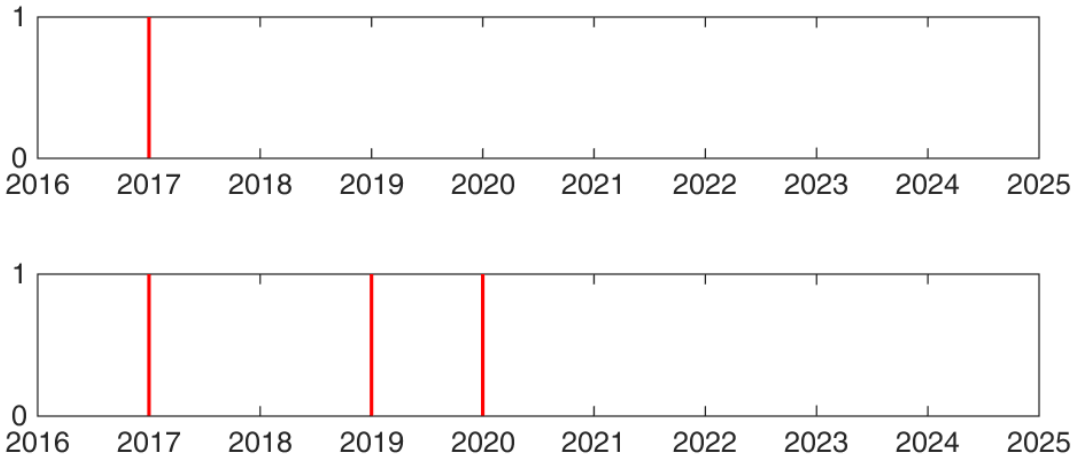

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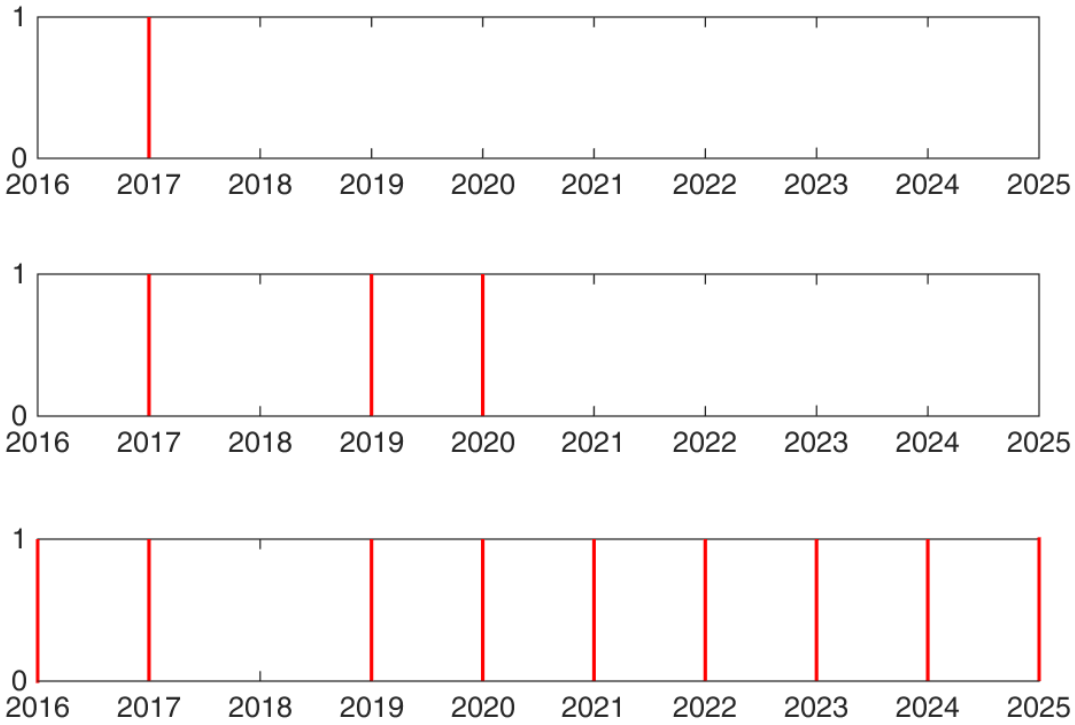

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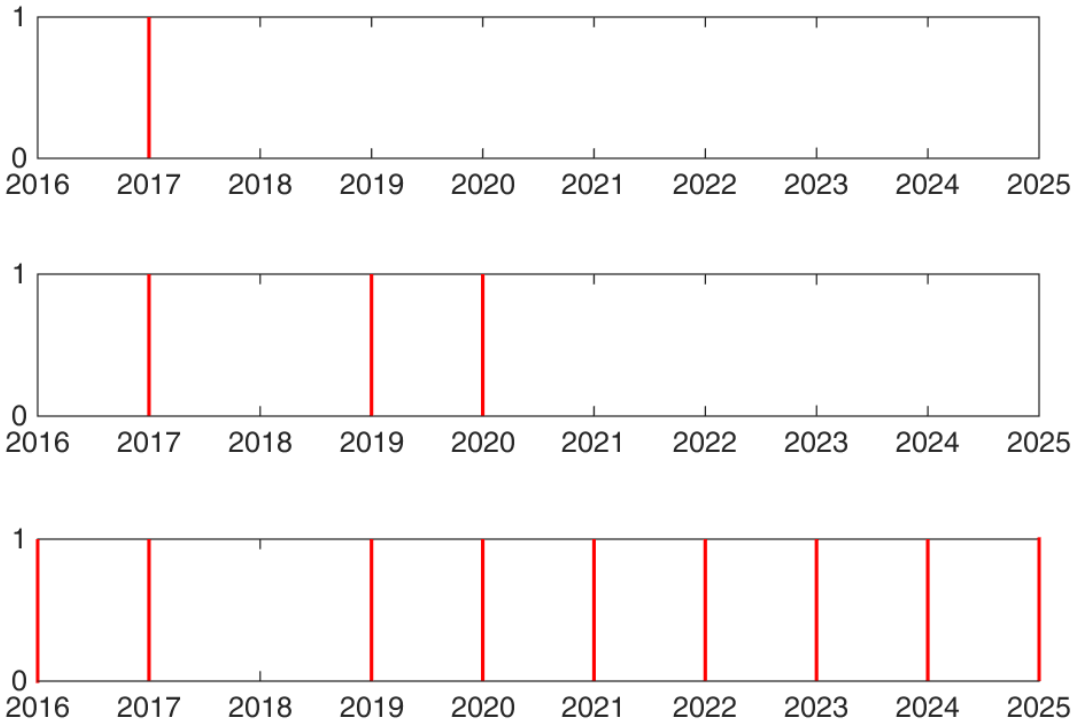
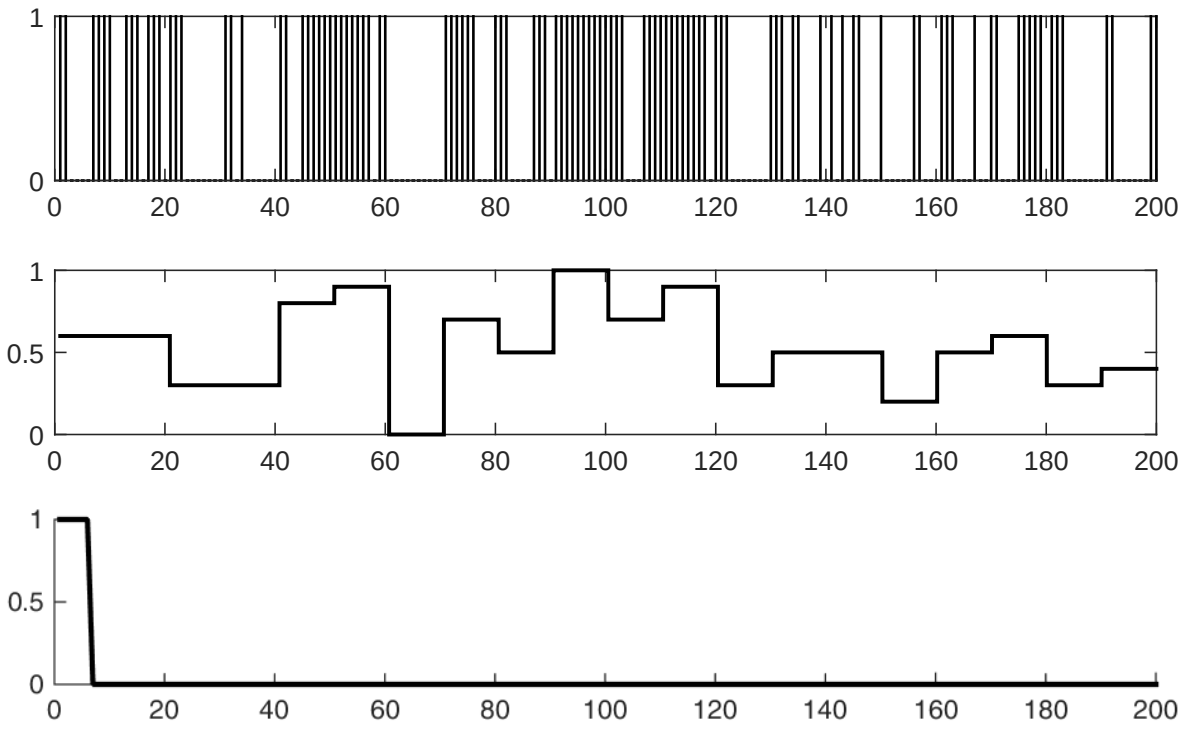


☞ Quantitative semantics measures the percentage of satisfaction of the signal within the associated window

Motivation (ctd.)



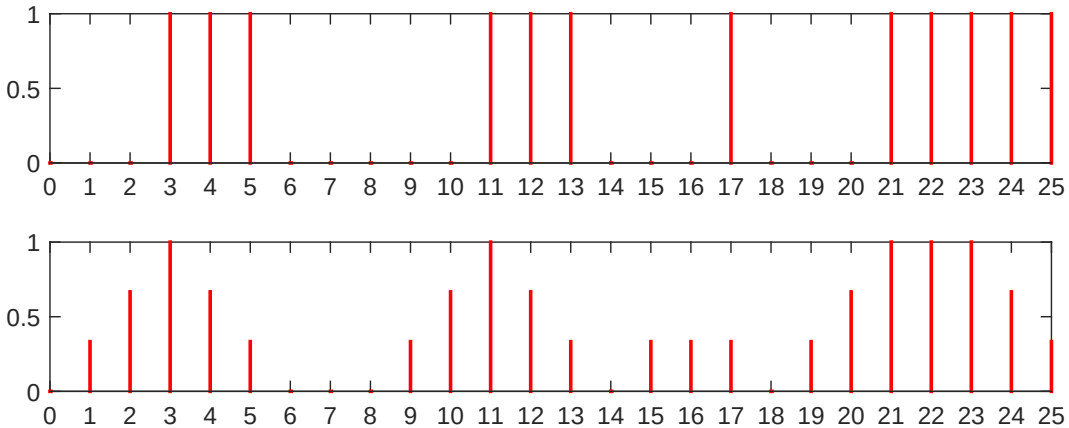
Spike sequence

Spike rate (Binning)

Spike rate (Window sliding)

$$conv(t) = \sum_{i=0}^{T} \rho(i) w(t-i).$$

MTL: Quantitative Semantics



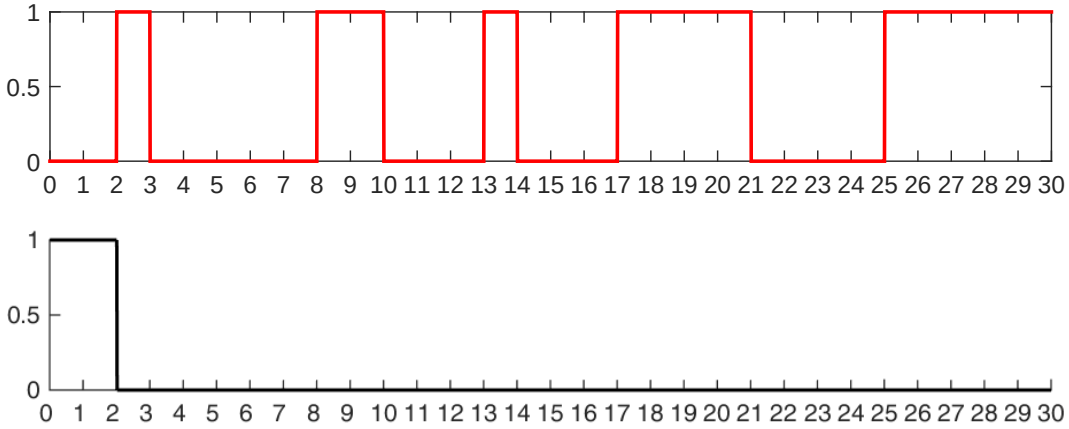
Discrete signal:
$(x, t) \models p$

Finally:
$[\![x, \Diamond_{[0,\,2]} p]\!](t)$

$$[\![x, \Diamond_I \varphi]\!](t) \;=\; \sum_{j \in \mathbb{T}} [\![x, \varphi]\!](j) \cdot w_I^+(t - j)$$

☞ Quantitative semantics measures the percentage of satisfaction of the signal within the associated window

MTL: Quantitative Semantics



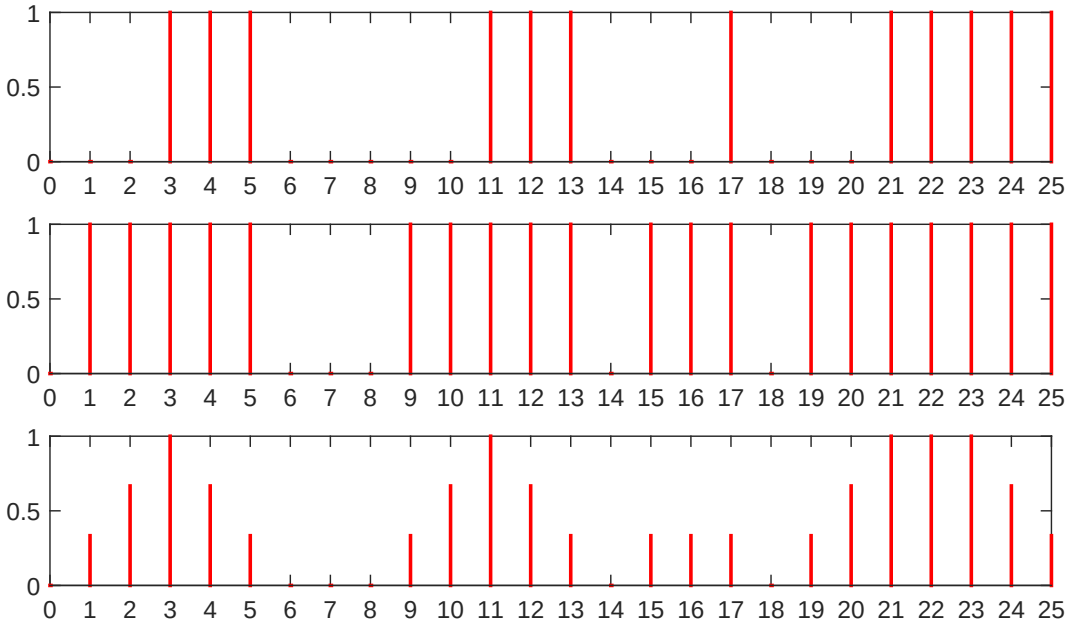
Continuous signal:
$(x, t) \models p$

Finally:
$[\![x, \Diamond_{[0,\,2]} p]\!](t)$

$$[\![x, \Diamond_I \varphi]\!](t) \;=\; \int_{\mathbb{T}} [\![x, \varphi]\!](j) \cdot w_I^+(t-j)\, d\tau$$

☞ Quantitative semantics measures the percentage of satisfaction of the signal within the associated window

MTL: Standard and Quantitative Semantics



Discrete signal:
$(x, t) \models p$

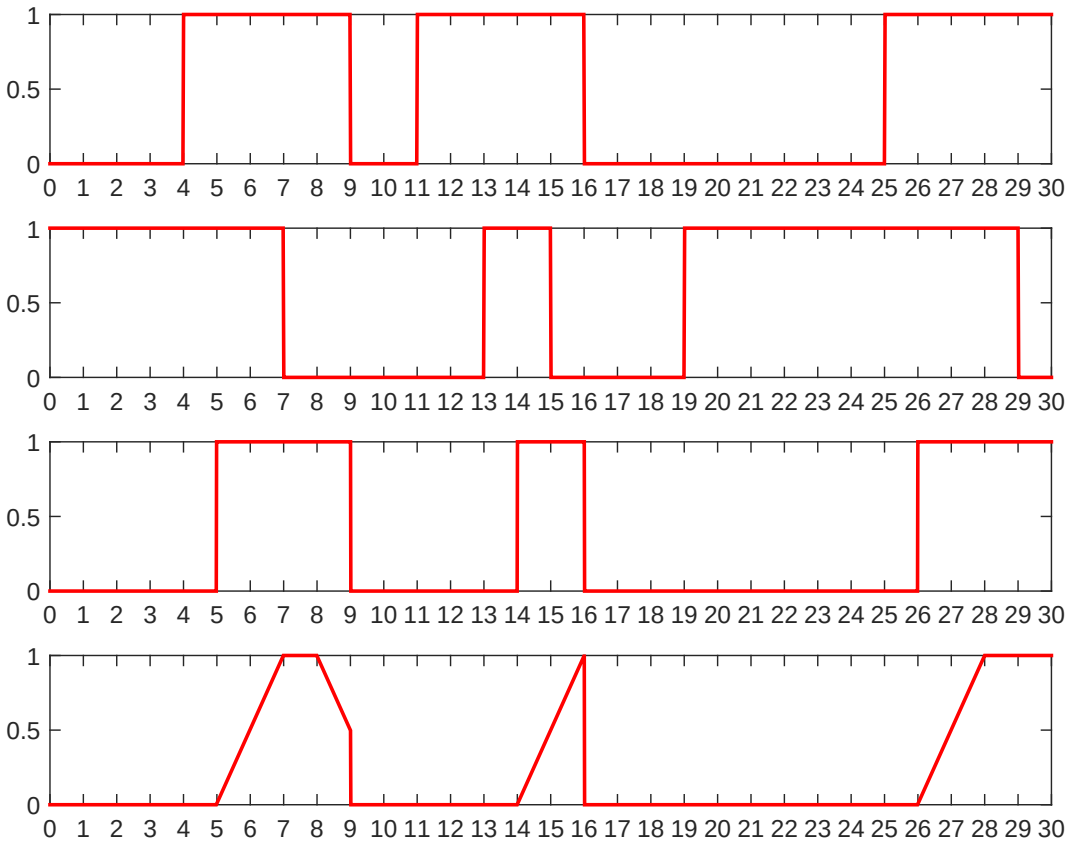
$(x, t) \models \Diamond_{[0,2]} p$

$[\![x, \Diamond_{[0,\,2]} p]\!](t)$

Theorem (Soundness). If φ is a positive normal form MTL formula and x is a discrete-time signal, then x satisfies φ, if and only if, its quantitative semantics is strictly greater than 0. It does not satisfy φ if its quantitative semantics is 0:

$$(x, t) \models \varphi \iff [\![x, \varphi]\!](t) > 0$$
$$(x, t) \not\models \varphi \iff [\![x, \varphi]\!](t) = 0$$

MTL: Standard and Quantitative Semantics



$(x,t) \models p$

$(x,t) \models q$

$(x,t) \models p\, S_{[1,3]}\, q$

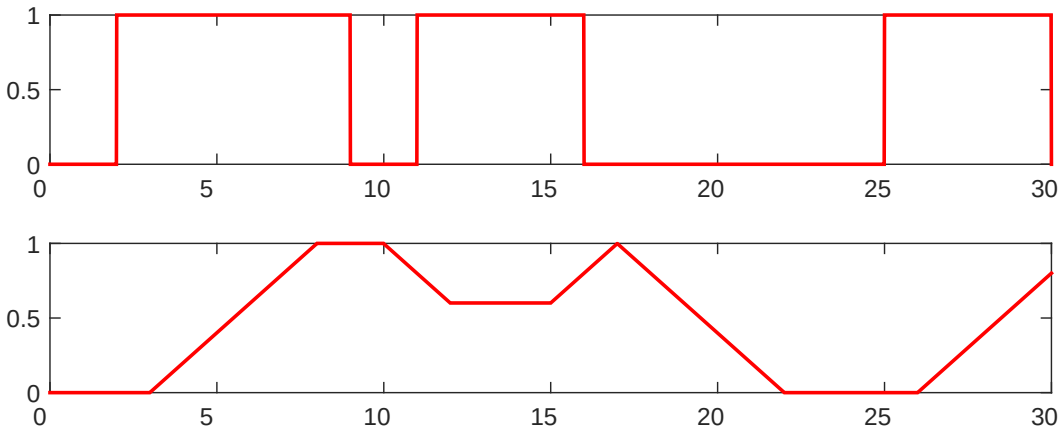
$[\![x,\, p\, S_{[1,3]}\, q]\!](t)$

Theorem (Soundness). Let φ be a positive normal form MTL formula and x be a continuous-time signal. Then the following properties hold:

$$[\![x,\varphi]\!](t) > 0 \quad \Longrightarrow \quad (x,t) \models \varphi$$
$$(x,t) \not\models \varphi \quad \Longrightarrow \quad [\![x,\varphi]\!](t) = 0$$

MTL: Smooth Evaluation

Example. Evaluation of MTL property $\varphi = \diamondminus_{[1,6]} p$.



$(x, t) \models p$

Rectangular kernel

$(x, t) \models \varphi$

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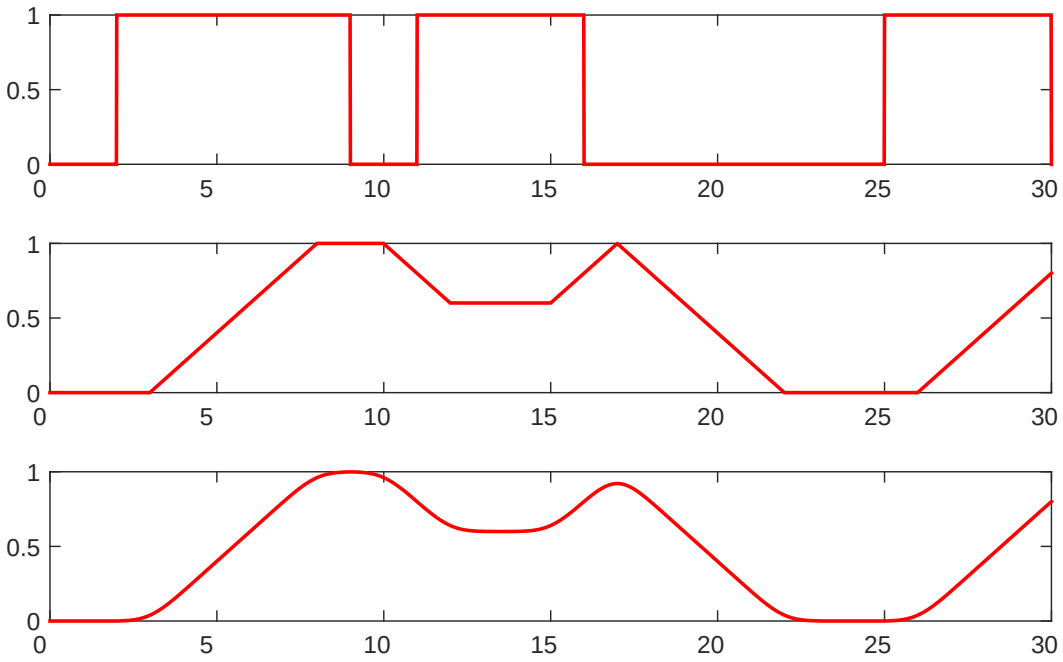

$(x, t) \models p$

Rectangular kernel
$(x, t) \models \varphi$

Sigmoidal kernel
$(x, t) \models \varphi$

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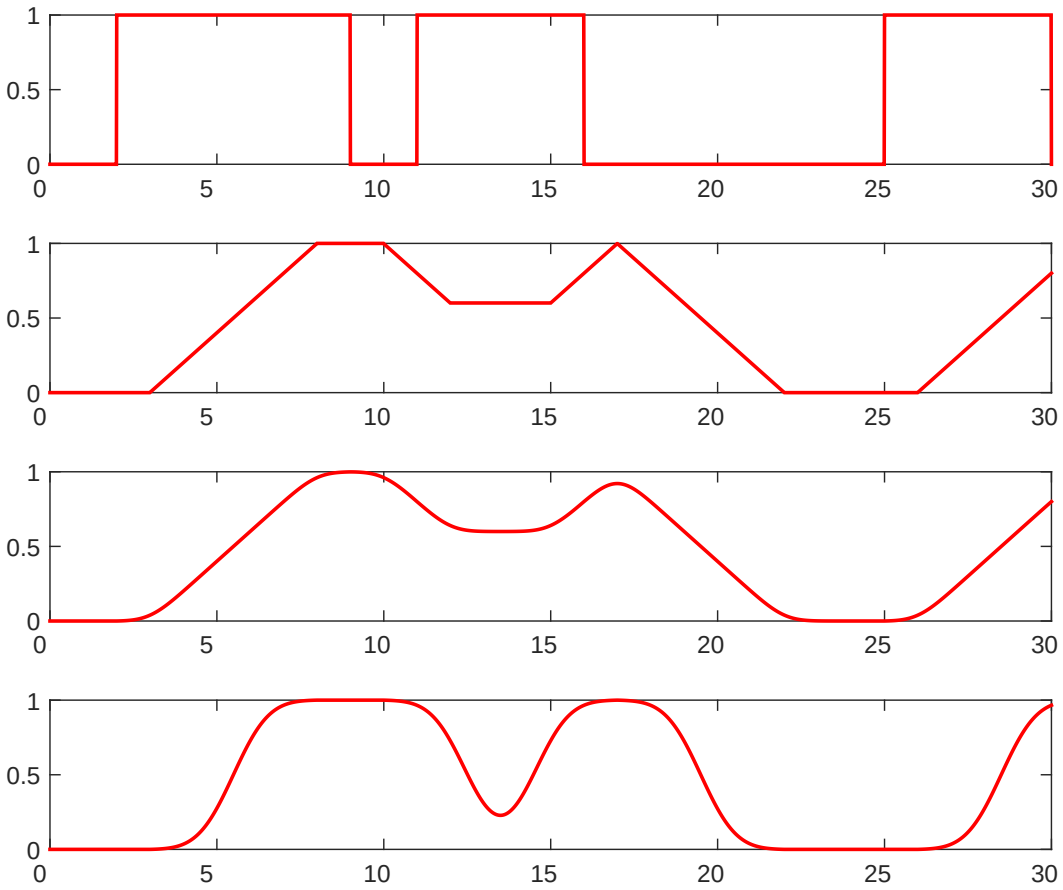

$(x,t) \models p$

Rectangular kernel
$(x,t) \models \varphi$

Sigmoidal kernel
$(x,t) \models \varphi$

Gaussian kernel
$(x,t) \models \varphi$

MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

$$conv(t) = (\rho * w)(t) = \sum_{i=0}^{T} \rho(i) w(t-i)$$

$\Longrightarrow$ *Quantitative* semantics

MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

$$conv(t) = (\rho * w)(t) = \sum_{i=0}^{T} \rho(i) w(t-i)$$

$\Longrightarrow$ *Quantitative* semantics

- Interpretation over $(\mathbb{B},$

MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

$$conv(t) = (\rho * w)(t) = \sum_{i=0}^{T} \rho(i) w(t-i)$$

$\Longrightarrow$ *Quantitative* semantics

- Interpretation over $(\mathbb{B}, \max,$

MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

$$conv(t) = (\rho * w)(t) = \sum_{i=0}^{T} \rho(i) w(t-i)$$

$\Longrightarrow$ *Quantitative* semantics

- Interpretation over $(\mathbb{B}, \max, \min, 0, 1)$

MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

$$conv(t) = (\rho * w)(t) = \sum_{i=0}^{T} \rho(i) w(t-i)$$

$\Longrightarrow$ *Quantitative* semantics

- Interpretation over $(\mathbb{B}, \max, \min, 0, 1)$

$$conv(t) = (\rho * w)(t) =$$

MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

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$\Longrightarrow$ *Quantitative* semantics

- Interpretation over $(\mathbb{B}, \max, \min, 0, 1)$

$$conv(t) = (\rho * w)(t) = \max_{i \in \mathbb{T}}$$

MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

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$\Longrightarrow$ *Quantitative* semantics

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MTL: Qualitative Semantics

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$\Longrightarrow$ *Qualitative* semantics

MTL: Standard and Qualitative Semantics

Theorem (Soundness). For a given signal x and MTL formula φ, x satisfies φ in the LTI-filtering interpretation, if and only if, it satisfies φ in the classic interpretation:

$$(x,t) \models_L \varphi \quad \Longleftrightarrow \quad (x,t) \models_C \varphi$$

Conclusions

- Qualitative filtering semantics
  - MTL can be viewed as linear time-invariant filtering
- Quantitative filtering semantics
  - We provided a novel quantitative semantics for MTL
  - We proved that this semantics is sound
- Smooth Evaluation
  - Depending on the filter's kernel, one can associate various quantitative semantics to a given MTL formula

☞ We also presented a full implementation of this approach

Future Work

In **future work** we will:

- explore alternative quantitative semantics for MTL, which are more informative, in case when the MTL formula is not satisfied
- clarify the logical meaning of other types of LTI-filter kernels: Gaussian, Sigmoidal, LPF, HPF, BPF
- investigate how such correspondence between MTL and LTI- filters can be exploited in order to build very efficient MTL monitors, using digital signal processors (DSPs)

OPTIONAL PART

References

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G. E. Fainekos and G. J. Pappas.
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Real-Time Systems, 2(4):255–299, 1990.

MTL: Metric Temporal Logic

$$
\begin{array}{lcl}
(x,i) \models p & \iff & x_p[i] = 1 \\
(x,i) \models \neg\varphi & \iff & (x,i) \not\models \varphi \\
(x,i) \models \varphi \vee \psi & \iff & (x,i) \models \varphi \text{ or } (x,i) \models \psi \\
(x,i) \models \varphi \wedge \psi & \iff & (x,i) \models \varphi \text{ and } (x,i) \models \psi \\
(x,i) \models \varphi \mathcal{U}_I \psi & \iff & \exists j \in (i+I) \cap \mathbb{T} : (x,j) \models \psi \\
 & & \text{and } \forall k \in (i,\, j) :\ (x,k) \models \varphi \\
(x,i) \models \varphi S_I \psi & \iff & \exists j \in (i-I) \cap \mathbb{T} : (x,j) \models \psi \\
 & & \text{and } \forall k \in (j,\, i) :\ (x,k) \models \varphi. \\
(x,i) \models \Diamond_I \varphi & \iff & \exists j \in (i+I) \cap \mathbb{T} : (x,j) \models \varphi \\
(x,i) \models \Box_I \varphi & \iff & \forall j \in (i+I) \cap \mathbb{T} : (x,j) \models \varphi \\
(x,i) \models \diamondminus_I \varphi & \iff & \exists j \in (i-I) \cap \mathbb{T} : (x,j) \models \varphi \\
(x,i) \models \boxminus_I \varphi & \iff & \forall j \in (i-I) \cap \mathbb{T} : (x,j) \models \varphi.
\end{array}
$$

[4] R. Koymans, Specifying Real-Time Properties with Metric Temporal Logic, 1990

Dioid

Group

- Closed
- Associativity
- Neutral element
- Inverse element

Semiring:

- Commutative monoid (group, no additive inverse) over $+$
- Monoid over $*$
- Distributivity

Dioid = Indempotent Semiring ($a + a = a$) = "Double Monoid"

MTL: Discrete-time Quantitative Semantics

$$
\begin{aligned}
[\![x, p]\!][i] &= x_p[i] \\
[\![x, \neg p]\!][i] &= 1 - [\![x, p]\!][i] \\
[\![x, \varphi \vee \psi]\!][i] &= \max([\![x, \varphi]\!][i],\ [\![x, \psi]\!][i]) \\
[\![x, \varphi \wedge \psi]\!][i] &= \min([\![x, \varphi]\!][i],\ [\![x, \psi]\!][i]) \\
[\![x, \Diamond_I \varphi]\!][i] &= \sum_{j \in \mathbb{T}} [\![x, \varphi]\!][j] \cdot w_I^+[i-j] \\
[\![x, \diamondminus_I \varphi]\!][i] &= \sum_{j \in \mathbb{T}} [\![x, \varphi]\!][j] \cdot w_I^-[i-j] \\
[\![x, \Box_I \varphi]\!][i] &= \min_{j \in (i+I) \cap \mathbb{T}} [\![x, \varphi]\!][j] \\
[\![x, \boxminus_I \varphi]\!][i] &= \min_{j \in (i-I) \cap \mathbb{T}} [\![x, \varphi]\!][j] \\
[\![x, \varphi \mathcal{U}_I \psi]\!][i] &= \frac{1}{|I|} \sum_{j \in I} [\![x, \Box_{[1,j-1]} \varphi]\!][i] \cdot [\![x, \Diamond_{\{j\}} \psi]\!][i]
\end{aligned}
$$

MTL: Continuous-time Quantitative Semantics

$$
\begin{aligned}
[\![x, p]\!](i) &= x_p(i) \\
[\![x, \neg p]\!](i) &= 1 - [\![x, p]\!](i) \\
[\![x, \varphi \vee \psi]\!](i) &= \max([\![x, \varphi]\!](i),\ [\![x, \psi]\!](i)) \\
[\![x, \varphi \wedge \psi]\!](i) &= \min([\![x, \varphi]\!](i),\ [\![x, \psi]\!](i)) \\
[\![x, \Diamond_I \varphi]\!](i) &= \int_{\mathbb{T}} [\![x, \varphi]\!](j) \cdot w_I^+(i-j)\ \mathrm{d}j \\
[\![x, \diamondminus_I \varphi]\!](i) &= \int_{\mathbb{T}} [\![x, \varphi]\!](j) \cdot w_I^-(i-j)\ \mathrm{d}j \\
[\![x, \Box_I \varphi]\!](i) &= \inf_{j \in (i+I) \cap \mathbb{T}} [\![x, \varphi]\!](j) \\
[\![x, \boxminus_I \varphi]\!](i) &= \inf_{j \in (i-I) \cap \mathbb{T}} [\![x, \varphi]\!](j) \\
[\![x, \varphi \mathcal{U}_I \psi]\!](i) &= \frac{1}{|I|} \int_I [\![x, \Box_{(0,j)} \varphi]\!](i) \cdot [\![x, \Diamond_{\{j\}} \psi]\!](i) \mathrm{d}j
\end{aligned}
$$

MTL: Discrete Qualitative Semantics

$$
\begin{aligned}
(x,i) \models p \quad &\iff \quad x_p[i] \\
(x,i) \models \neg\varphi \quad &\iff \quad 1 - (\,(x,i) \models \varphi\,) \\
(x,i) \models \varphi \vee \psi \quad &\iff \quad \max\left((x,i) \models \varphi,\ (x,i) \models \psi\right) \\
(x,i) \models \Diamond_I \varphi \quad &\iff \quad \max_{j \in \mathbb{T}} \min\left((x,j) \models \varphi,\ w_I^+[i-j]\right) \\
(x,i) \models \Box_I \varphi \quad &\iff \quad 1 - (\,(x,i) \models \Diamond_I \neg\varphi\,) \\
(x,i) \models \diamondminus_I \varphi \quad &\iff \quad \max_{j \in \mathbb{T}} \min\left((x,j) \models \varphi,\ w_I^-[i-j]\right) \\
(x,i) \models \boxminus_I \varphi \quad &\iff \quad 1 - (\,(x,i) \models \diamondminus_I \neg\varphi\,) \\
(x,i) \models \varphi\,\mathcal{U}_I\,\psi \quad &\iff \quad \max_{j \in I} \min\Big((x,i) \models \Box_{[1,j-1]}\varphi,\ (x,i) \models \Diamond_{\{j\}}\psi\Big) \\
(x,i) \models \varphi\,S_I\,\psi \quad &\iff \quad \max_{j \in I} \min\Big((x,i) \models \boxminus_{[1,j-1]}\varphi,\ (x,i) \models \diamondminus_{\{j\}}\psi\Big).
\end{aligned}
$$

MTL: Continuous Qualitative Semantics

$$
\begin{aligned}
(x,i) \models p \quad &\iff \quad x_p(i) \\
(x,i) \models \neg\varphi \quad &\iff \quad 1 - (\,(x,i) \models \varphi\,) \\
(x,i) \models \varphi \vee \psi \quad &\iff \quad \max\left((x,i) \models \varphi,\ (x,i) \models \psi\right) \\
(x,i) \models \Diamond_I \varphi \quad &\iff \quad \sup_{j \in \mathbb{T}} \min\left((x,j) \models \varphi,\ w_I^+(i-j)\right) \\
(x,i) \models \Box_I \varphi \quad &\iff \quad 1 - (\,(x,i) \models \Diamond_I \neg\varphi\,) \\
(x,i) \models \diamondminus_I \varphi \quad &\iff \quad \sup_{j \in \mathbb{T}} \min\left((x,j) \models \varphi,\ w_I^-(i-j)\right) \\
(x,i) \models \boxminus_I \varphi \quad &\iff \quad 1 - (\,(x,i) \models \diamondminus_I \neg\varphi\,) \\
(x,i) \models \varphi\,\mathcal{U}_I\,\psi \quad &\iff \quad \sup_{j \in I} \min\Big((x,i) \models \Box_{(0,j)}\varphi,\ (x,i) \models \Diamond_{\{j\}}\psi\Big) \\
(x,i) \models \varphi\,S_I\,\psi \quad &\iff \quad \sup_{j \in I} \min\Big((x,i) \models \boxminus_{(0,j)}\varphi,\ (x,i) \models \diamondminus_{\{j\}}\psi\Big).
\end{aligned}
$$

MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

$$conv(t) = (\rho * w)(t) = \begin{cases} \sum_{i=0}^{T} \rho(i) w(t-i) \\ \\ \int_{0}^{T} \rho(\tau) w(t-\tau) d\tau \end{cases}$$

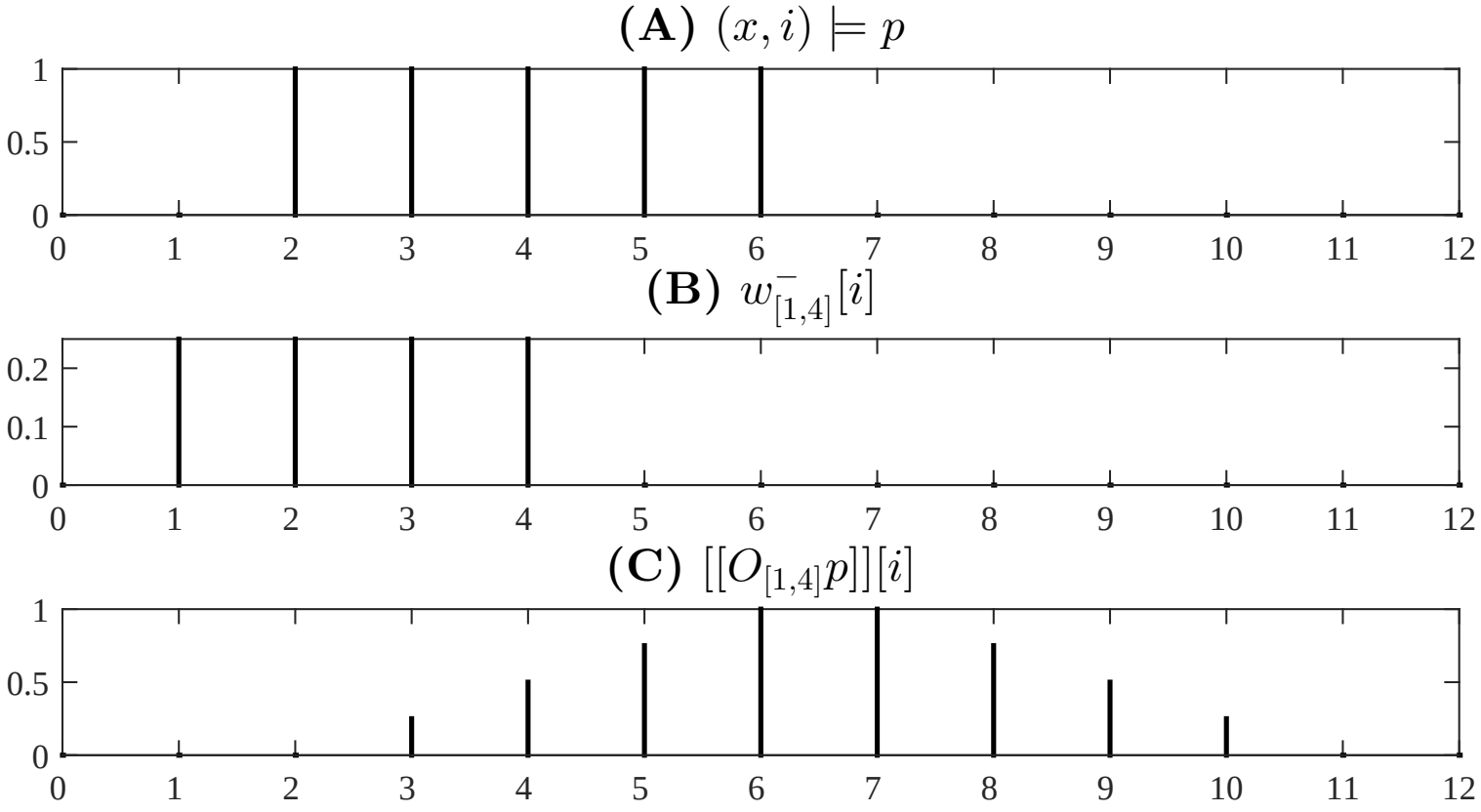
$\Longrightarrow$ *Quantitative* semantics

- Interpretation over $(\{0,1\}, \max, \min, 0, 1)$

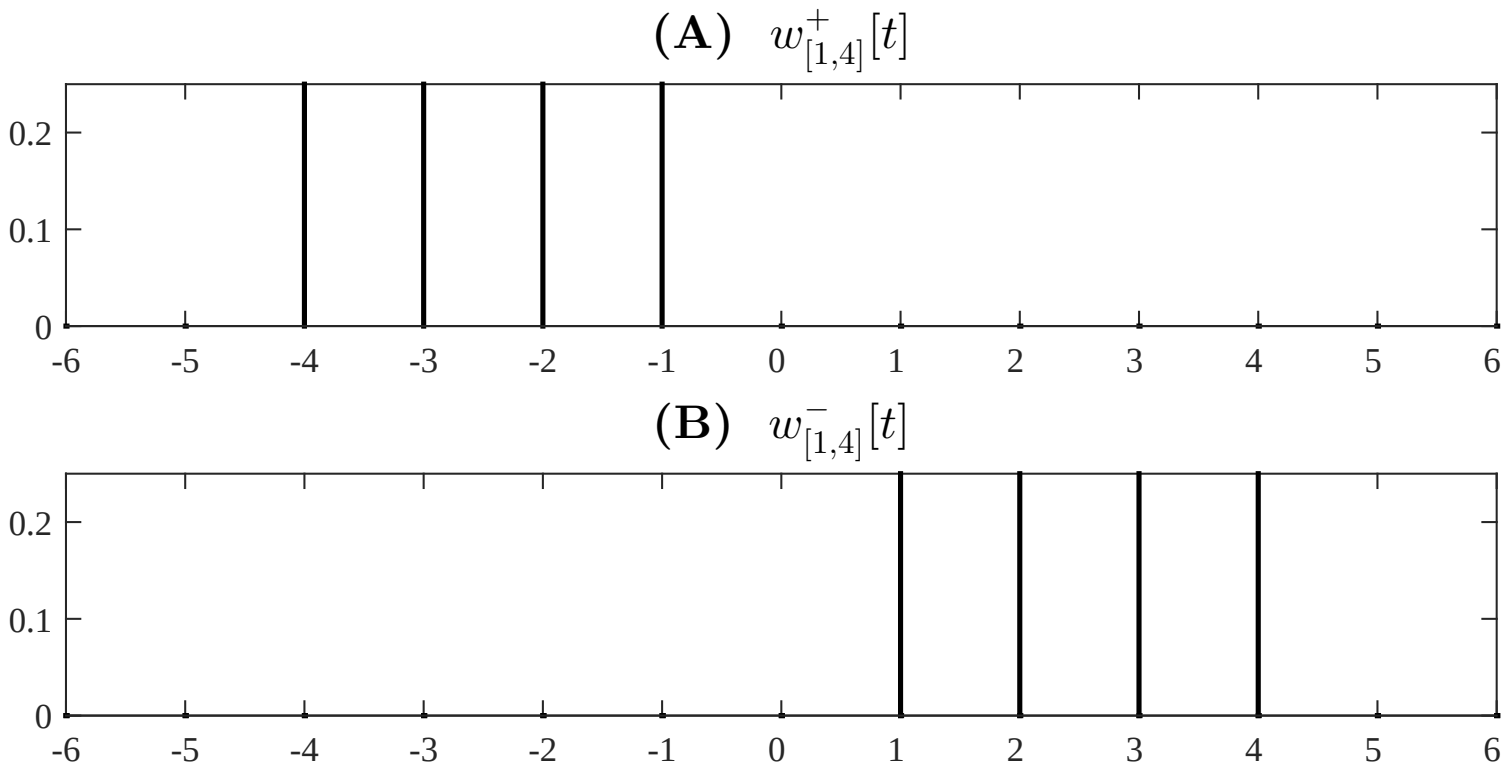
$$conv(t) = (\rho * w)(t) = \begin{cases} \max_{i \in \mathbb{T}} \min\left(\rho(i),\ w(t-i)\right) \\ \\ \sup_{i \in \mathbb{T}} \min\left(\rho(i),\ w(t-i)\right) \end{cases}$$

$\Longrightarrow$ *Qualitative* semantics

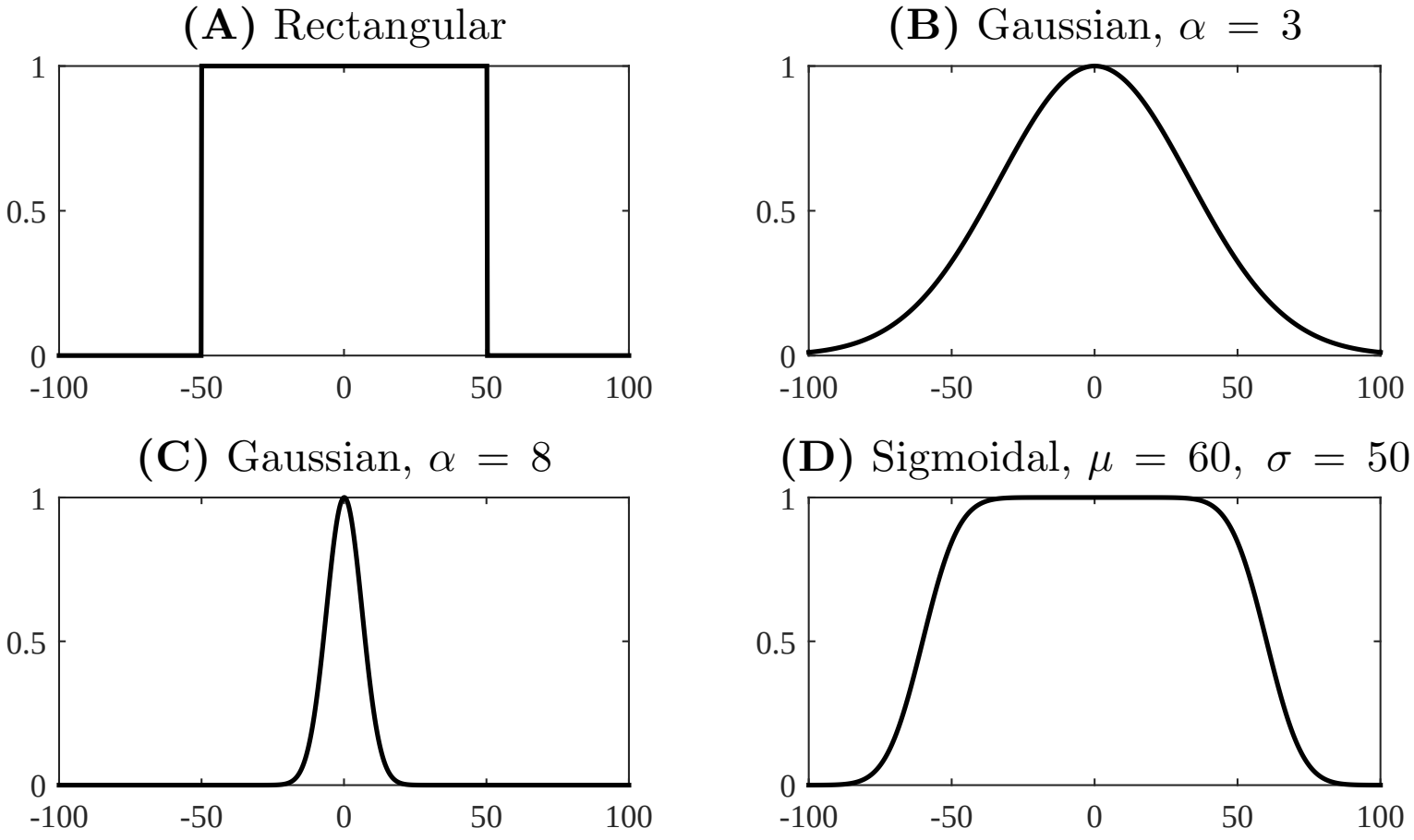
MTL: Rectangular kernel



MTL: Rectangular kernel

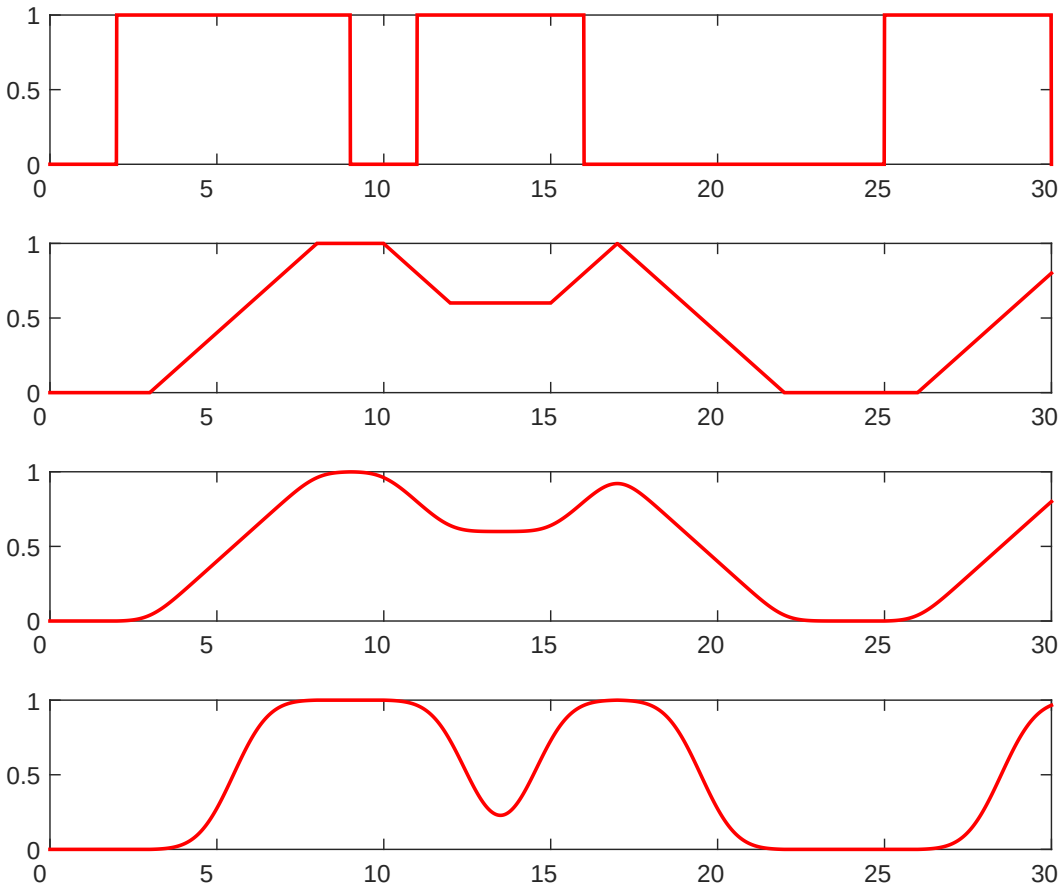


MTL: Smooth Evaluation



MTL: Smooth Evaluation

Example. Evaluation of MTL property $\diamondminus_{[1,6]}$.



$(x, t) \models p$

Rectangular kernel

Sigmoidal kernel

Gaussian kernel