Temporal Logic as Filtering
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Motivation

Figure: Neuron recordings

Motivation

Spike sequence
Motivation

Spike sequence

Spike rate (Binning)
Motivation

Spike sequence

Spike rate (Binning)

Spike rate (Window sliding)
Motivation

Spike sequence

Spike rate (Binning)

Spike rate (Window sliding)
Filtering: Convolution

\[
\text{conv}(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i),
\]

\[
w(t) = \begin{cases} 
\frac{1}{\Delta t}, & \text{if } t \in \left[ -\frac{\Delta t}{2}; \frac{\Delta t}{2} \right] \\
0, & \text{otherwise}
\end{cases}
\]

**Figure:** Convolution with a rectangular window
MTL: Metric Temporal Logic

MTL Syntax

\[ \Phi = \{ p \mid \neg \varphi \mid \varphi \lor \psi \mid \varphi \mathcal{U}_I \psi \mid \varphi \mathcal{S}_I \psi \} \]

- \( \varphi \land \psi = \neg (\neg \varphi \lor \neg \psi) \)
- Finally operator: \( \square_I \varphi = \top \mathcal{U}_I \varphi \)
- Globally operator: \( \lozenge_I \varphi = \neg \square_I \neg \varphi \)
- Analogously for past operators Once \( \lozenge_I \) and Historically \( \square_I \)

Example. \((x, 0) \models \lozenge_{[0,2]} p\)
MTL: Metric Temporal Logic

Example. \((x, 0) \models \Diamond [0,2]p\)

Example. \((x, 0) \models \Diamond [2,4]p \land \Box [3,6]q\)
MTL: Standard Semantics

Discrete signal:

\[(x, t) \models p\]

Finally:

\[(x, t) \models \Diamond [0,2]p\]

\[(x, t) \models \Diamond_I \varphi \iff \exists j \in (t + I) \cap T : (x, j) \models \varphi\]
MTL: Standard Semantics

Discrete signal:
\[(x, t) \models p\]

Finally:
\[(x, t) \models \diamond [0,2]p\]

\[(x, t) \models \diamond_I \varphi \iff \exists j \in (t + I) \cap \mathbb{T} : (x, j) \models \varphi\]

- Standard Boolean MTL semantics is qualitative
- It does not provide any quantitative measure
MTL: Robust Semantics

$\varphi =$ “If the value of the signal drops below -1, then it should also raise above +1 within two time units.”

MTL: Robust Semantics

\[ \varphi = \text{"If the value of the signal drops below -1, then it should also raise above +1 within two time units."} \]

\[ (p_1 \rightarrow \Diamond_{\leq 2} p_2) \]

MTL: Robust Semantics

\[ \varphi = \text{“If the value of the signal drops below -1, then it should also raise above +1 within two time units.”} \]

\[ \square (p_1 \rightarrow \Diamond_{\leq 2} p_2) \]

Robustness degree is the bound on the perturbation that the signal can tolerate without changing the truth value of a specification.

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MTL: Time Robustness

\[ \varphi = \lozenge_{[1, 3]}(s \geq 1) \]

- Time robustness indicates the effect on satisfaction of shifting events in time.

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Time robustness indicates the effect on satisfaction of shifting events in time.

MTL: Quantitative Semantics

Example. “In a next 10 years You will finally get a Turing award...”
MTL: Quantitative Semantics

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\[ \varphi = \Diamond_{[0,9]} TuringAward \]
MTL: Quantitative Semantics

Example. “In a next 10 years You will finally get a Turing award…”

$$\varphi = \Diamond_{[0,9]} Turing\text{Award}$$
Example. “In the next 10 years, you will finally get a Turing award...”

\[ \varphi = \Diamond_{[0,9]} \text{TuringAward} \]
MTL: Quantitative Semantics

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Example. “In a next 10 years You will finally get a Turing award…”

$$\varphi = \Diamond_{[0,9]} TuringAward$$

Quantitative semantics measures the percentage of satisfaction of the signal within the associated window.
Motivation (ctd.)

Spike sequence

Spike rate (Binning)

Spike rate (Window sliding)

\[ \text{conv}(t) = \sum_{i=0}^{T} \rho(i)w(t-i). \]
MTL: Quantitative Semantics

Discrete signal: 
\((x, t) \models p\)

Finally:
\(\llbracket x, \Diamond [0, 2]p \rrbracket(t)\)

Quantitative semantics measures the percentage of satisfaction of the signal within the associated window.
Continuous signal: 

\[(x, t) \models p\]

Finally:

\[[x, \Diamond_{[0, 2]} p](t)\]

Quantitative semantics measures the percentage of satisfaction of the signal within the associated window.
MTL: Standard and Quantitative Semantics

Discrete signal:

\((x, t) \models p\)

\((x, t) \models \Diamond_{[0,2]} p\)

\([x, \Diamond_{[0,2]} p](t)\)

**Theorem (Soundness).** If \(\varphi\) is a positive normal form MTL formula and \(x\) is a discrete-time signal, then \(x\) satisfies \(\varphi\), if and only if, its quantitative semantics is strictly greater than 0. It does not satisfy \(\varphi\) if its quantitative semantics is 0:

\[(x, t) \models \varphi \iff [x, \varphi](t) > 0\]

\[(x, t) \not\models \varphi \iff [x, \varphi](t) = 0\]
Theorem (Soundness). Let $\varphi$ be a positive normal form MTL formula and $x$ be a continuous-time signal. Then the following properties hold:

$$[x, \varphi](t) > 0 \implies (x, t) \models \varphi$$

$$ (x, t) \not\models \varphi \implies [x, \varphi](t) = 0$$
Example. Evaluation of MTL property $\varphi = \bigotimes_{[1, 6]} p$.

\[ (x, t) \models p \]

Rectangular kernel
\[ (x, t) \models \varphi \]
Example. Evaluation of MTL property $\varphi = \diamondsuit_{[1,6]} p$.

$(x, t) \models p$

Rectangular kernel
$(x, t) \models \varphi$

Sigmoidal kernel
$(x, t) \models \varphi$
MTL: Smooth Evaluation

Example. Evaluation of MTL property $\varphi = \bigotimes [1, 6] p$.

$(x, t) \models p$

Rectangular kernel
$(x, t) \models \varphi$

Sigmoidal kernel
$(x, t) \models \varphi$

Gaussian kernel
$(x, t) \models \varphi$
MTL: Qualitative Semantics

- Interpretation over \((\mathbb{R}, +, \cdot, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i)
\]

\[\implies \text{Quantitative semantics}\]
MTL: Qualitative Semantics

- Interpretation over \((\mathbb{R}, +, \cdot, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i)
\]

\(\implies\) Quantitative semantics

- Interpretation over \((\mathbb{B},\),
MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

$$\text{conv}(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i)$$

$\implies$ Quantitative semantics

- Interpretation over $(\mathbb{B}, \max, \cdot)$
MTL: Qualitative Semantics

- Interpretation over $(\mathbb{R}, +, \cdot, 0, 1)$

$$\text{conv}(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i)$$

$$\implies \text{Quantitative semantics}$$

- Interpretation over $(\mathbb{B}, \max, \min, 0, 1)$
MTL: Qualitative Semantics

- **Interpretation over** \((\mathbb{R}, +, \cdot, 0, 1)\)

\[
conv(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i)
\]

\(\implies\) **Quantitative semantics**

- **Interpretation over** \((\mathbb{B}, \max, \min, 0, 1)\)

\[
conv(t) = (\rho \ast w)(t) =
\]
MTL: Qualitative Semantics

➤ Interpretation over \((\mathbb{R}, +, \cdot, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i)
\]

⇒ Quantitative semantics

➤ Interpretation over \((\mathbb{B}, \max, \min, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \max_{i \in T} w(t-i)
\]
MTL: Qualitative Semantics

- Interpretation over \((\mathbb{R}, +, \cdot, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i)
\]

\(\Rightarrow\) Quantitative semantics

- Interpretation over \((\mathbb{B}, \max, \min, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \max_{i \in \mathbb{T}} \min
\]
MTL: Qualitative Semantics

- Interpretation over \((\mathbb{R}, +, \cdot, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i)
\]

\[\implies\] Quantitative semantics

- Interpretation over \((\mathbb{B}, \max, \min, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \max_{i \in T} \min (\rho(i), w(t-i))
\]
MTL: Qualitative Semantics

- Interpretation over \((\mathbb{R}, +, \cdot, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \sum_{i=0}^{T} \rho(i)w(t-i)
\]

\(\implies\) **Quantitative semantics**

- Interpretation over \((\mathbb{B}, \max, \min, 0, 1)\)

\[
\text{conv}(t) = (\rho \ast w)(t) = \max_{i \in \mathbb{T}} \min \rho(i), w(t-i)
\]

\(\implies\) **Qualitative semantics**
Theorem (Soundness). For a given signal \( x \) and MTL formula \( \varphi \), \( x \) satisfies \( \varphi \) in the LTi-filtering interpretation, if and only if, it satisfies \( \varphi \) in the classic interpretation:

\[
(x, t) \models_L \varphi \iff (x, t) \models_C \varphi
\]
Conclusions

- Qualitative filtering semantics
  - MTL can be viewed as linear time-invariant filtering

- Quantitative filtering semantics
  - We provided a novel quantitative semantics for MTL
  - We proved that this semantics is sound

- Smooth Evaluation
  - Depending on the filter’s kernel, one can associate various quantitative semantics to a given MTL formula

- We also presented a full implementation of this approach
In future work we will:

- explore alternative quantitative semantics for MTL, which are more informative, in case when the MTL formula is not satisfied
- clarify the logical meaning of other types of LTI-filter kernels: Gaussian, Sigmoidal, LPF, HPF, BPF
- investigate how such correspondence between MTL and LTI-filters can be exploited in order to build very efficient MTL monitors, using digital signal processors (DSPs)
OPTIONAL PART
References

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MTL: Metric Temporal Logic

\[(x, i) \models p \iff x_p[i] = 1\]
\[(x, i) \models \neg \varphi \iff (x, i) \not\models \varphi\]
\[(x, i) \models \varphi \lor \psi \iff (x, i) \models \varphi \text{ or } (x, i) \models \psi\]
\[(x, i) \models \varphi \land \psi \iff (x, i) \models \varphi \text{ and } (x, i) \models \psi\]
\[(x, i) \models \varphi \mathcal{U} I\psi \iff \exists j \in (i + I) \cap T : (x, j) \models \psi \text{ and } \forall k \in (i, j) : (x, k) \models \varphi\]
\[(x, i) \models \varphi \mathcal{S} I\psi \iff \exists j \in (i - I) \cap T : (x, j) \models \psi \text{ and } \forall k \in (j, i) : (x, k) \models \varphi\]
\[(x, i) \models \Diamond I\varphi \iff \exists j \in (i + I) \cap T : (x, j) \models \varphi\]
\[(x, i) \models \Box I\varphi \iff \forall j \in (i + I) \cap T : (x, j) \models \varphi\]
\[(x, i) \models \Diamond I\varphi \iff \exists j \in (i - I) \cap T : (x, j) \models \varphi\]
\[(x, i) \models \Box I\varphi \iff \forall j \in (i - I) \cap T : (x, j) \models \varphi\]

Dioid

Group
- Closed
- Associativity
- Neutral element
- Inverse element

Semiring:
- Commutative monoid (group, no additive inverse) over $\oplus$
- Monoid over $\ast$
- Distributivity

Dioid = Indempotent Semiring ($a + a = a$) = “Double Monoid”
MTL: Discrete-time Quantitative Semantics

\[
\begin{align*}
[x, p][i] & = x_p[i] \\
[x, \neg p][i] & = 1 - [x, p][i] \\
[x, \varphi \lor \psi][i] & = \max([x, \varphi][i], [x, \psi][i]) \\
[x, \varphi \land \psi][i] & = \min([x, \varphi][i], [x, \psi][i]) \\
[x, \Diamond_I \varphi][i] & = \sum_{j \in T} [x, \varphi][j] \cdot w^+_I[i - j] \\
[x, \lozenge_I \varphi][i] & = \sum_{j \in T} [x, \varphi][j] \cdot w^-_I[i - j] \\
[x, \Box_I \varphi][i] & = \min_{j \in (i+I) \cap \mathbb{T}} [x, \varphi][j] \\
[x, \square_I \varphi][i] & = \min_{j \in (i-I) \cap \mathbb{T}} [x, \varphi][j] \\
[x, \varphi U_I \psi][i] & = \frac{1}{|I|} \sum_{j \in I} [x, \Box[1,j-1] \varphi][i] \cdot [x, \Diamond \{j\} \psi][i]
\end{align*}
\]
MTL: Continuous-time Quantitative Semantics

\[
\begin{align*}
[x, p](i) &= x_p(i) \\
[x, \neg p](i) &= 1 - [x, p](i) \\
[x, \varphi \lor \psi](i) &= \max([x, \varphi](i), [x, \psi](i)) \\
[x, \varphi \land \psi](i) &= \min([x, \varphi](i), [x, \psi](i)) \\
[x, \Diamond_I \varphi](i) &= \int_{T} [x, \varphi](j) \cdot w^+(i-j) \, dj \\
[x, \Diamond_I \varphi](i) &= \int_{T} [x, \varphi](j) \cdot w^-(i-j) \, dj \\
[x, \Box_I \varphi](i) &= \inf_{j \in (i+I) \cap T} [x, \varphi](j) \\
[x, \Box_I \varphi](i) &= \inf_{j \in (i-I) \cap T} [x, \varphi](j) \\
[x, \varphi \mathcal{U}_I \psi](i) &= \frac{1}{|I|} \int_{I} [x, \Box_{(0,j)} \varphi](i) \cdot [x, \Diamond_{\{j\}} \psi](i) \, dj
\end{align*}
\]
MTL: Discrete Qualitative Semantics

\[(x, i) \models p \iff x_p[i]\]
\[(x, i) \models \neg \varphi \iff 1 - ( (x, i) \models \varphi )\]
\[(x, i) \models \varphi \lor \psi \iff \max ((x, i) \models \varphi, (x, i) \models \psi)\]
\[(x, i) \models \diamond_I \varphi \iff \max \min \ ( (x, j) \models \varphi, w_I^+[i-j])\]
\[(x, i) \models \square_I \varphi \iff 1 - ( (x, i) \models \diamond_I \neg \varphi )\]
\[(x, i) \models \square_I \varphi \iff 1 - ( (x, i) \models \diamond_I \neg \varphi )\]
\[(x, i) \models \varphi \mathcal{U}_I \psi \iff \max \min \left((x, i) \models \square_{[1,j-1]} \varphi, (x, i) \models \diamond \{j\} \psi \right)\]
\[(x, i) \models \varphi \mathcal{S}_I \psi \iff \max \min \left((x, i) \models \square_{[1,j-1]} \varphi, (x, i) \models \diamond \{j\} \psi \right).\]
MTL: Continuous Qualitative Semantics

\[(x, i) \models p \iff x_p(i)\]
\[(x, i) \models \neg \varphi \iff 1 - ((x, i) \models \varphi)\]
\[(x, i) \models \varphi \lor \psi \iff \max \left( (x, i) \models \varphi, (x, i) \models \psi \right)\]
\[(x, i) \models \Box_I \varphi \iff \sup \min \left( (x, j) \models \varphi, w_I^+(i - j) \right)\]
\[(x, i) \models \Diamond_I \varphi \iff 1 - ((x, i) \models \Box_I \neg \varphi)\]
\[(x, i) \models \Box_I \varphi \iff 1 - ((x, i) \models \Diamond_I \neg \varphi)\]
\[(x, i) \models \varphi U_I \psi \iff \sup \min \left( (x, i) \models \Box_{(0,j)} \varphi, (x, i) \models \Diamond \{j\} \psi \right)\]
\[(x, i) \models \varphi S_I \psi \iff \sup \min \left( (x, i) \models \Box_{(0,j)} \varphi, (x, i) \models \Diamond \{j\} \psi \right).\]
MTL: Qualitative Semantics

- Interpretation over \((\mathbb{R}, +, \cdot, 0, 1)\)

\[
conv(t) = (\rho \ast w)(t) = \begin{cases} \sum_{i=0}^{T} \rho(i)w(t-i) \\ \int_{0}^{T} \rho(\tau)w(t-\tau)d\tau \end{cases}
\]

\[\implies \text{Quantitative semantics}\]

- Interpretation over \((\{0, 1\}, \max, \min, 0, 1)\)

\[
conv(t) = (\rho \ast w)(t) = \begin{cases} \max \min_{i \in \mathbb{T}} (\rho(i), w(t-i)) \\ \sup \min_{i \in \mathbb{T}} (\rho(i), w(t-i)) \end{cases}
\]

\[\implies \text{Qualitative semantics}\]
MTL: Rectangular kernel

(A) \((x, i) \models p\)

(B) \(w_{[1,4]}^{-}[i]\)

(C) \([O_{[1,4]}p][i]\)
MTL: Rectangular kernel

(A) $w^{+}_{[1,4]}[t]$

(B) $w^{-}_{[1,4]}[t]$

\[
\begin{align*}
\text{(A)} & \quad w^{+}_{[1,4]}[t] \\
\text{(B)} & \quad w^{-}_{[1,4]}[t]
\end{align*}
\]
MTL: Smooth Evaluation

(A) Rectangular

(B) Gaussian, $\alpha = 3$

(C) Gaussian, $\alpha = 8$

(D) Sigmoidal, $\mu = 60$, $\sigma = 50$
**MTL: Smooth Evaluation**

**Example.** Evaluation of MTL property $\diamondsuit[1, 6]$.

\[(x, t) \models p\]

- **Rectangular kernel**
- **Sigmoidal kernel**
- **Gaussian kernel**